Math 113 Lecture #29
§11.5: Alternating Series

The Alternating Series Test. After defining what is means for a series to converge, we have mainly focused on developing tools to determine convergence or divergence for series with positive terms.

We now turn our attention to developing a tool to deal with a series whose terms alternate in sign, such as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

Theorem. If $\sum b_n$ is a series with positive terms for which

(i) $b_{n+1} \leq b_n$ for all n, and

(ii) $\lim_{n\to\infty} b_n = 0$,

then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

converges.

Remark. What is remarkable about this Theorem is that NO assumption about the convergence or divergence of $\sum b_n$ is made. It could be that $\sum b_n$ diverges, and the conclusion of this Theorem stills holds.

Remark. Let us consider what the alternating series in this Theorem is doing before proving the Theorem.

The partial sums are

$$s_1 = b_1,\ s_2 = b_1 - b_2,\ s_3 = b_1 - b_2 + b_3,\ s_4 = b_1 - b_2 + b_3 - b_4,\ \text{etc.}$$

The assumptions that $b_n \geq 0$ and $b_{n+1} \leq b_n$ for all n implies that

$$s_1 \geq s_2,\ s_2 \leq s_3,\ s_3 \geq s_4,\ \text{etc.}$$

More importantly,

$$s_2 \leq s_4 \leq s_6 \leq \cdots \text{ and } s_1 \geq s_3 \geq s_5 \geq \cdots.$$

The even partial sums are an nondecreasing sequence (i.e., can go up or stay the same, but never go down), and the odd partial sums are a nonincreasing sequence (i.e., can go down or stay the same, but never go up).

It seems like the even partitions and the odd partial sums ought to converge to something in between them.

Proof of the Alternating Series Test. We more carefully show that the even partial sums are a nondecreasing sequence that is bounded above:

$$\begin{aligned} s_2 = b_1 - b_2 &\geq 0 \text{ since } b_2 \leq b_1, \\ s_4 = s_2 + (b_3 - b_4) &\geq s_2 \text{ since } b_4 \leq b_3, \\ s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) &\geq s_{2n-2} \text{ since } b_{2n} \leq b_{2n-1}. \end{aligned}$$

Thus we have

$$0 \leq s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n} \leq \cdots$$

where

$$s_{2n} = b_1 - (b_2 - b_3) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}.$$

Since $b_{n+1} \leq b_n$ for all n, every term of the form $b_{2n-2} - b_{2n-1}$ is nonnegative, i.e., bigger or equal to 0.

This gives us an upper bound the even partial sums:

$$s_{2n} \leq b_1.$$

By the "obvious" variation of the Monotonic Sequence Theorem, the bounded above nondecreasing sequence $\{s_{2n}\}$ converges to say s.

So all we have to do now is check the sequence of odd partial sums converges to s too:

$$\begin{aligned} \lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} \left(s_{2n} + b_{n+1}\right) \\ &= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{n+1} \\ &= s + 0 \\ &= s, \end{aligned}$$

where we have used the assumption that $b_n \to 0$ as $n \to \infty$. □

Example 1. Test the following alternating series for convergence:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \ln n}{n}.$$

We need to check the two conditions on the terms $b_n = (\ln n)/n$.

We show that $\{b_n\}$ is eventually decreasing using calculus:

$$f(x) = \frac{\ln x}{x} \quad \Rightarrow \quad f'(x) = \frac{1 - \ln x}{x^2} \leq 0 \text{ for } x \geq e.$$

Thus the function $f(x)$ is nonincreasing once x is bigger than e.

Since $b_n = f(n)$, the sequence $\{b_n\}$ is nonincreasing, i.e., $b_{n+1} \leq b_n$, for all $n \geq 3$.

We compute the limit of $\{b_n\}$ by computing the limit of $f(x)$ as $x \to \infty$ by L'Hospital's Rule:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Thus $\{b_n\}$ converges to 0, and the Alternating Series Test shows that the series converges.

The Alternating Series Estimate. The alternating nature of a convergence alternating series lends itself to an *easy* estimate of the remainder of any partial sum s_n from the value s of the convergence series:

$$R_n = s - s_n.$$

Theorem. Let $\{b_n\}$ be a sequence for which $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$. If $s = \sum(-1)^{n-1}b_n$, then

$$|R_n| \leq b_{n+1}.$$

Proof. We know from the proof of the Alternating Series Test that the value s lies between any two consecutive partial sums:

$$s_n \leq s \leq s_{n+1} \text{ for } n \text{ even, or } s_{n+1} \leq s \leq s_n \text{ for } n \text{ odd.}$$

From these inequalities we get some more:

$$0 \leq s - s_n \leq s_{n+1} - s_n \text{ for even } n, \text{ and } s_{n+1} - s_n \leq s - s_n \leq 0 \text{ for odd } n.$$

Interpreting these as absolute values we get

$$|s - s_n| \leq |s_{n+1} - s_n| \text{ for even } n \text{ and } |s - s_n| \leq |s_{n+1} - s_n| \text{ for odd } n.$$

Thus we get for all n the inequalities

$$|s - s_n| \leq |s_{n+1} - s_n|.$$

However s_{n+1} and s_n differ by $\pm b_{n+1}$, so that $|s - s_n| \leq b_{n+1}$. □

For convergent alternating series, estimating the error is relatively easy.

Example 2. The alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n5^n} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n}$$

is convergent by the Alternating Series Test because $b_n = (n5^n)^{-1}$ is nonincreasing and has limit 0.

To get $|R_n|$ smaller than 0.0001 requires that

$$b_{n+1} \leq 0.0001.$$

This is an inequality not readily solved for n:

$$\frac{1}{(n+1)5^{n+1}} \leq 0.0001.$$

But computing the left hand side on a calculator for $n = 2$, $n = 3$, and $n = 4$ gives

$$b_{2+1} \approx 0.0026, b_{3+1} \approx 0.0004, \text{ and } b_{4+1} \approx 0.000064 < 0.0001.$$

So the partial sum s_4 is within 0.0001 of the actual value of the series.