Math 113 Lecture #30
§11.6: Absolute Convergence and the Ratio and Root Tests

Absolute Convergence. The Integral Test and the Comparison Tests apply to series with positive terms, and the Alternating Series Test applies to series the sign of whose terms alternate regularly.

But how do we test the convergence or divergence of a series in which the terms are positive and negative without any regularity in the switching of the sign?

The answer is to consider a related series in which all the terms are positive.

Definitions. A series $\sum a_n$ is called **absolute convergent** if the series $\sum |a_n|$ converges.

A series $\sum a_n$ that is convergent but not absolutely convergent is called **conditionally convergent**.

Conditionally convergent series like alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

are quite bizarre in that the order of the terms can be rearranged to sum to any real number.

Absolutely convergent series like

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

always sum to the same thing no matter how the order of the terms are rearranged.

Here is another nice property of absolute convergent series.

Theorem. If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.

Proof. Since either $|a_n|$ is either a_n or $-a_n$, then

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$

Assuming that $\sum a_n$ is absolutely convergent implies that $\sum 2|a_n|$ is convergent.

The Comparison Test applies to show that $\sum(a_n + |a_n|)$ is convergent too.

Since $a_n = a_n + |a_n| - |a_n|$, then

$$\sum a_n = \sum \big(a_n + |a_n|\big) - \sum |a_n|$$

which shows that $\sum a_n$ is the sum of two convergent series, and is convergent too. $\square$

Example 1. Is the series $\displaystyle\sum \frac{\sin(4n)}{4^n}$ convergent?

The terms of this series change signs but not regularly like an alternating series.

We consider whether or not the series is absolutely convergent:

$$\sum \frac{|\sin(4n)|}{4^n}.$$

Since $|\sin(4n)| \leq 1$, we have

$$\frac{|\sin(4n)|}{4^n} \leq \frac{1}{4^n}.$$

Since $\sum 1/4^n$ converges (it is a geometric series for $r = 1/4$), the Comparison Test shows that $\sum |\sin(4n)|/4^n$ converges.

This means that the original series absolutely converges, and hence converges too.

The Ratio Test.

Since absolute convergence implies convergence, it would be great if we can test for absolute convergence directly.

The Ratio Test provides one way to do this.

Theorem. For a series $\sum a_n$ with nonzero terms, set

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Assume that L exists, or that $L = \infty$.

(i) If $L < 1$, then $\sum a_n$ converges absolutely.

(ii) If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.

Remark. If $L = 1$, then nothing can be said about the convergence or divergence of $\sum a_n$. Another test has to be employ to determine convergence or divergence.

Proof. For absolute convergence, we will compare a convergence geometric series with $\sum a_n$.

Since $L < 1$ there is a constant r such that $L < r < 1$.

Since $|a_{n+1}/a_n| \to L$ there is N such that $|a_{n+1}/a_n| < r$ for all $n \geq N$.

In particular, this means that

$$\begin{aligned} |a_{N+1}| &\leq |a_N| r, \\ |a_{N+2}| &\leq |a_{N+1}| r = |a_N| r^2, \\ |a_{N+3}| &\leq |a_{N+2}| r = |a_N| r^3, \end{aligned}$$

and so forth, giving for $k \geq 1$ the inequalities

$$|a_{N+k}| \leq |a_N| r^k.$$

The series

$$\sum_{k=1}^{\infty} |a_N| r^k$$

is a convergent geometric series since $|r| < 1$.

The Comparison Test now applies to show that

$$\sum_{n=N+1}^{\infty} |a_n|$$

converges too (the first N terms do not affect the convergence of the series).

Thus $\sum a_n$ is absolutely convergent.

Now assume that $L > 1$ or $L = \infty$.

Since $|a_{n+1}/a_n| \to L > 1$, there is N such that $|a_{n+1}/a_n| > 1$ for all $n \geq N$.

This means that $|a_{n+1}| > |a_n|$ for all $n \geq N$, and this prevents the a_n's from going to 0:

$$\lim_{n\to\infty} a_n \neq 0, \text{ or does not exist..}$$

By the Test for Divergence, the series $\sum a_n$ diverges. □

Example 2. Does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 2^n}{n!}$ converge?

We apply the Ratio Test to this:

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+2}(n+1)^2 2^{n+1}/(n+1)!}{(-1)^{n+1} n^2 2^n/n!} \right| \\
&= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \frac{2^{n+1}}{2^n} \frac{n!}{(n+1)!} \\
&= 2 \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^2 \frac{1}{n+1} \\
&= 0.
\end{aligned}$$

Since L exists and is smaller than 1, the Ratio Test shows that the series converges absolutely, and hence converges.

The Root Test.

The Ratio Test applies nicely when there are factorials in the series.

When there are powers, sometimes the following test may determine convergence or divergence.

Theorem For $\sum a_n$ with nonzero terms, set

$$L = \lim_{n\to\infty} |a_n|^{1/n}.$$

Assume that L exists or is ∞.

(i) If $L < 1$, then $\sum a_n$ converges absolutely.

(ii) If $L > 1$ or $L = \infty$, then $\sum a_n$ diverges.

Remark. If $L = 1$, then we have to determine convergence or divergence some other way.

Example 3. Does $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$ converge?

We apply the Root Test:

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(-1)^n}{(\ln n)^n} \right|^{1/n} \\ &= \lim_{n\to\infty} \frac{1}{(\ln n)} \\ &= 0. \end{aligned}$$

Thus the series converges absolutely.