Math 113 Lecture #31
§11.7: Strategy for Testing Series

Summary Review of Series and Convergence Tests. A series is an infinite sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

The sequence of partial sums attached to a series is

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

A series is said to be convergent if its sequence of partial sums converges:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n.$$

Otherwise, the series is divergent, i.e., $\lim_{n\to\infty} s_n$ does not exist.

A series is absolutely convergent if

$$\sum_{n=1}^{\infty} |a_n| \text{ converges.}$$

An absolutely convergent series is a convergent series.

A series is conditionally convergent if it converges but is not absolutely convergent.

Divergence Test. If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

Integral test. Suppose there is a continuous, positive, decreasing function f defined on $[1,\infty)$ such that $a_n = f(n)$. The series $\sum_{n=1}^{\infty} a_n$ converges if and only if

$$\int_1^{\infty} f(t)\ dt \text{ converges.}$$

Remainder Estimate for the Integral Test. If $s = \sum_{n=1}^{\infty} a_n$ converges, then

$$s_n + \int_{n+1}^{\infty} f(t)\ dt \leq s \leq s_n + \int_n^{\infty} f(t)\ dt.$$

The midpoint of this interval in which s lives is another estimate of s, with an error no bigger than half the length of the interval.

Comparison Tests. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\sum b_n$ converges and $a_n \leq b_n$ for all n, then $\sum a_n$ converges. If $\sum b_n$ diverges and $a_n \geq b_n$ for all n, then $\sum a_n$ diverges.

Comparison Test: Estimating Sums. If $s = \sum a_n$ with positive terms converges by comparison with the convergent $t = \sum b_n$ with positive terms, then the remainder $R_n = s - s_n$ for the series $\sum a_n$ is bounded above the the remainder $T_n = t - t_n$ for the series $\sum b_n$.

Limit Comparison Test. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}$$

exists and is a positive number, then either $\sum a_n$ and $\sum b_n$ both converge, or $\sum a_n$ and $\sum b_n$ both diverge.

If c exists and equals 0, then convergence of $\sum b_n$ implies convergence of $\sum a_n$.

If $c = \infty$, then divergence of $\sum b_n$ implies divergence of $\sum a_n$.

Alternating Series Test. If $\sum b_n$ is a series with positive terms such that $b_{n+1} \leq b_n$ for all n and $\lim_{n\to\infty} b_n = 0$, then the alternating series $\sum(-1)^{n+1}b_n$ converges.

Clue when to use: the presence of $(-1)^n$ or $(-1)^{n+1}$ in the terms.

Alternating Series Estimate. If $s = \sum(-1)^{n+1}b_n$ is a convergent alternating series, then

$$|s - s_n| \leq b_{n+1}.$$

Ratio Test for Absolute Convergence. Suppose $\sum a_n$ has nonzero terms and set

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $L < 1$, then $\sum a_n$ is absolutely convergent. If $L > 1$ then $\sum a_n$ is divergent. If $L = 1$, the Ratio Test fails.

Clue when to use: the presence of factorials in the terms.

Root Test for Absolute Convergence. Suppose $\sum a_n$ has nonzero terms and set

$$L = \lim_{n\to\infty} |a_n|^{1/n}.$$

If $L < 1$, then $\sum a_n$ converges absolutely. If $L > 1$, then $\sum a_n$ diverges. If $L = 1$, then the Root Test fails.

Clue when to use: the presence of n^{th} powers in the terms.

Known Series. Here are series and their convergence or divergence you should know.

Geometric Series. For a nonzero constant a and a constant r, this is the series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots.$$

This series converges if and only if $|r| < 1$ and in this case there is a formula for its sum:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

p-Series. For a constant p, this is the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

This series converges if $p > 1$ and diverges if $p \leq 1$. The Harmonic series is the p-series with $p = 1$.

Telescoping Series. For a positive integer k and a function f defined on $[1, \infty)$, this is a series of the from

$$\sum_{n=1}^{\infty} \big(f(n) - f(n+k)\big).$$

A telescoping series converges if $\lim_{n\to\infty} f(n) = 0$ because the limit of

$$s_n = f(1) + f(2) + \cdots + f(k) + \cdots - f(n+1) - f(n+2) - \cdots - f(n+k)$$

is

$$f(1) + f(2) + \cdots + f(k)$$

which is the value of the convergent telescoping series.

Practice. The key to successfully determining if a series converges or diverges is practice, practice, and more practice.

Here is a selection of series for which we will determine absolute convergence, conditional convergence, or divergence. If a series is convergent, find either the value of the series or a good estimate.

$$\sum_{n=1}^{\infty} \frac{1}{n^2 5^n},$$

$$\sum_{n=1}^{\infty} \frac{2}{n^2 + 2n},$$

$$\sum_{n=1}^{\infty} \frac{1}{n + 3^n},$$

$$\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}},$$

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n} - 1},$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}},$$

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}},$$

$$\sum_{n=1}^{\infty} (2^{1/n} - 1)^n.$$