Math 113 Lecture #32
§11.8: Power Series

What is a Power Series? We are all familiar with polynomials:

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n.$$

A power series is like a polynomial (whose degree is finite) except that it has infinite degree:

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots = \sum_{n=0}^{\infty} c_n x^n.$$

We can shift the graph of a polynomial by composing it with $x - a$, and do likewise for a power series:

$$\sum_{n=0}^{\infty} c_n (x - a)^n.$$

The constants $c_0, c_1, c_2, \ldots$ are called the **coefficients** of the power series, and the constant a is called the **center** of the power series.

The Radius of Convergence. For what values of x does a power series converge and for what values of x does it diverge? Are these values of x in intervals, or more sporadic than that?

You might think that the answer of only three possibilities is quite surprising.

Theorem. A power series $\displaystyle\sum_{n=0}^{\infty} c_n (x - a)^n$ converges

(i) only for $x = a$ (i.e., only at the center) and diverges everywhere else,

(ii) for all x, or

(iii) on an open interval $|x - a| < R$ for some $R > 0$ and diverges on $|x - a| > R$.

In the third possibility, the number R is called the **radius of convergence**, and the interval $|x - a| < R$ the **open interval of convergence**.

What is not specified in the Theorem in the third case is what happens at the endpoints of $|x - a| < R$.

We have a convergence or divergence test to determine the what happens at the endpoints.

It might happen that the power series converges at one endpoint, so that that endpoint would be included in the interval of convergence.

Or it might happen that the power series does not converge at an endpoint, so that that endpoint does not belong to the interval of convergence.

That gives 4 types of the **interval of convergence** for a power series when its radius of convergence R is positive but finite:

$$|x - a| < R, \quad -R \le x - a < R, \quad R < x - a \le R, \quad |x - a| \le R.$$

How do we find R? By the Ratio or Root Tests, as illustrated next.

Example 1. Find the interval of convergence for $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{4^n \ln n}$.

If we let $b_n = \dfrac{(-1)^n x^n}{4^n \ln n}$, then we can apply the Ratio Test to the series $\sum b_n$:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| \\
&= \lim_{n\to\infty} \frac{|x^{n+1}|/4^{n+1}\ln(n+1)}{|x^n|/4^n \ln n} \\
&= \frac{|x|}{4} \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \\
&= \frac{|x|}{4} \lim_{n\to\infty} \frac{1/n}{1/(n+1)} \\
&= \frac{|x|}{4}.
\end{aligned}
$$

We need $L < 1$ for convergence, and this gives us the radius of convergence:

$$
\frac{|x|}{4} < 1 \quad \Rightarrow \quad |x| < 4.
$$

This gives us the open interval of convergence, but to get the interval of convergence we investigate the convergence or divergence of the power series at the endpoints.

At the endpoint $x = 4$, the power series becomes

$$
\sum_{n=2}^{\infty} \frac{(-1)^n 4^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}.
$$

This is an alternating series and so converges by the Alternating Series Test.

Thus the endpoint $x = 4$ is part of the interval of convergence.

At the endpoint $x = -4$, the power series becomes

$$
\sum_{n=2}^{\infty} \frac{(-1)^n(-4)^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}.
$$

This series diverges by comparison with $\sum(1/n)$ because $n \geq \ln n$, i.e., $1/n \leq 1/\ln n$.

Thus the endpoint $x = -4$ is not part of the interval of convergence.

The interval of convergence for the power series is

$$
-4 < x \leq 4.
$$

Example 2. The Bessel function of order 0 is defined by the power series

$$
J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = 1 - \frac{x^2}{4} + \frac{x^4}{64} + \cdots.
$$

The function $y = J_0(x)$ is "formally" a solution of the differential equation,

$$x^2 y'' + xy' + x^2 y = 0$$

which appears in engineering and physics.

It is true that a power series with a positive radius of convergence R is differentiable with a derivative given by convergent power series with the same radius of convergence R (and we will be considering this at a later time).

We find the radius of convergence for $J_0(x)$, we apply the Ratio Test: with
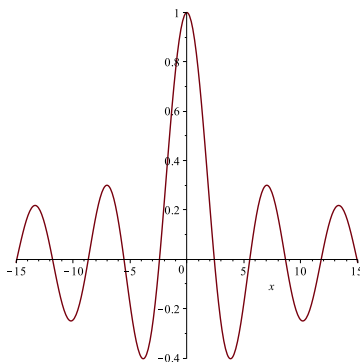
$$a_n = (-1)^n x^{2n}/2^{2n}(n!)^2$$

we have

$$
\begin{aligned}
L &= \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}} \right| \\
&= \lim_{n \to \infty} \frac{x^2}{4(n+1)^2} \\
&= 0.
\end{aligned}
$$

Thus $L = 0$ for any x, and since $L < 1$, we have that $R = \infty$.

With $R = \infty$ there are no endpoints to check, and the interval of convergence for $J_0(x)$ is $(-\infty, \infty)$.

Here is the graph of $J_0(x)$ in case you wanted to see what its graph looks like.

Example 3. Find the interval of convergence for the power series

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{x+5}{2} \right)^n.$$

We rewrite the terms of the power series so we can identity the center of the power series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (x+5)^n,$$

which gives the center of the power series as $a = -5$.

We use the Ratio Test to find the radius of convergence: with

$$a_n = \frac{(-1)^n}{2^n}(x+5)^n,$$

we have

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x+5)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(-1)^n(x+5)^n} \right| \\ &= \lim_{n\to\infty} \frac{|x+5|}{2} \\ &= \frac{|x+5|}{2}; \end{aligned}$$

by setting $L < 1$ we obtain the radius of convergence:

$$\frac{|x+5|}{2} < 1 \;\Rightarrow\; |x+5| < 2,$$

so that $R = 2$, and the open interval of convergence is

$$-2 < x + 5 < 2 \text{ or } -7 < x < -3.$$

We have the two endpoints of the open interval of convergence to check for convergence.

At the endpoint $x = -7$, we have the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(-2)^n = \sum_{n=0}^{\infty} 1^n.$$

The terms of this series do not go to zero, and therefore the series diverges by the Divergence Test.

At the endpoint $x = -3$, we have the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} 2^n = \sum_{n=0}^{\infty} (-1)^n.$$

The terms of this series do not go to zero, and therefore the series diverges by the Divergence Test.

The interval of convergence for the power series is $-7 < x < -3$.

Example 4. What is the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}?$$

We use the Ratio Test to find the radius of convergence; with

$$a_n = \frac{(-1)^n x^{2n+1}}{2n+1},$$

we have

$$
\begin{aligned}
L &= \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{2(n+1)+1} \cdot \frac{2n+1}{(-1)^n x^{2n+1}} \right| \\
&= \lim_{n \to \infty} x^2 \cdot \frac{2n+1}{2(n+1)+1} \\
&= x^2;
\end{aligned}
$$

setting $L < 1$ gives $x^2 < 1$ or $|x| < 1$, so the radius of convergence is $R = 1$.

We have two endpoints to check.

At the endpoint $x = 1$ the power series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n 1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

This alternating series converges by the Alternating Series Test because the terms $1/(2n+1)$ go to 0 monotonically.

This alternating series converges to $\pi/4$, so that

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots .$$

At the endpoint $x = -1$ the power series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}.$$

This alternating series converges by the Alternating Series Test because the terms $1/(2n+1)$ go to 0 monotonically.

So the interval of convergence for the power series is $-1 \le x \le 1$.

Example 5. Find the interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

We use the Ratio Test to find the radius of convergence; with

$$a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

we have

$$L = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$$
$$= \lim_{n \to \infty} x^2 \cdot \frac{1}{(2n+2)(2n+3)}$$
$$= 0;$$

this value of L is always smaller than 1 for any x, so the radius of convergence is $R = \infty$, and with no endpoints to check, the interval of convergence for the power series is $-\infty < x < \infty$.

Example 6. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{8^n n! (n+1)!} (x-11)^n.$$

We find the radius of convergence by the Ratio Test; for

$$a_n = \frac{(-1)^n (2n)!}{8^n n! (n+1)!} (x-11)^n$$

we have

$$L = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} (2(n+1))! (x-11)^{n+1}}{8^{n+1} (n+1)! (n+2)!} \cdot \frac{8^n n! (n+1)!}{(-1)^n (2n)! (x-11)^n} \right|$$
$$= \lim_{n \to \infty} \frac{|x-11|}{8} \cdot \frac{1}{(n+1)(n+2)} \cdot (2n+1)(2n+2)$$
$$= \frac{|x-11|}{8} \lim_{n \to \infty} \frac{2n+1}{n+1} \cdot \frac{2n+2}{n+2}$$
$$= \frac{|x-11|}{2};$$

setting $L < 1$ gives as the radius of convergence $R = 2$.