Math 113 Lecture #33
§11.9: Representations of Functions as Power Series

Finding Power Series Representations.
We will see how to use the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,$$

to represent or express certain functions as a power series. At first this may seem a strange thing to do to perfectly good functions.

Example 1. Represent by a power series the function

$$\frac{1}{1+x}.$$

To use the geometric series we need express the x in the form $-x$:

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + \cdots.$$

The radius of convergence of this power series is $|-x| < 1$ which is the same as $|x| < 1$.

Example 2. Represent by a power series the function

$$\frac{x}{2x^2+1}.$$

Here we leave the numerator outside for a minute and rewrite the $2x^2$ term as $-2x^2$:

$$\frac{x}{2x^2+1} = x\frac{1}{1-(-2x^2)} = x\sum_{n=0}^{\infty}(-2x^2)^n = x\sum_{n=0}^{\infty}(-2)^n x^{2n} = \sum_{n=0}^{\infty}(-2)^n x^{2n+1}.$$

Multiplying the series through by x is legitimate because it does not depend on n (think of it as multiplying through by a constant).

The radius of convergence of this power series is $|-2x^2| < 1$, or $|x| < \sqrt{2}/2$.

Differentiation and Integration of Power Series.
For a polynomial, we differentiate and integrate term by term, and this principle also holds for infinite-degree polynomials or power series.

Theorem. If the radius of convergence for $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is $R > 0$ or $R = \infty$, then $f(x)$ is differentiable and continuous on $(a-R, a+R)$ (the open interval of convergence), and

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty}\frac{d}{dx}c_n(x-a)^n = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1} = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

and

$$\int f(x)\ dx = C + \int \left(\sum_{n=0}^{\infty} c_n (x-a)^n \right)\ dx = C + \sum_{n=0}^{\infty} \int c_n (x-a)^n\ dx$$
$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1},$$

where C is the arbitrary constant of integration. The radii of convergence of the differentiated and integrated power series are both R.

Example 3. Find a power series representation for the function

$$\frac{x^2}{(1-2x)^2}.$$

We focus on the denominator first:

$$\frac{1}{(1-2x)^2} = \frac{1}{2}\frac{d}{dx}\left(\frac{1}{1-2x}\right).$$

We can write the term being differentiated here in terms of a geometric series:

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n,$$

where the radius of convergence is $|2x| < 1$, i.e., $|x| < 1/2$.

To this point we have

$$\frac{1}{(1-2x)^2} = \frac{1}{2}\frac{d}{dx}\sum_{n=0}^{\infty}(2x)^n = \frac{1}{2}\sum_{n=0}^{\infty} 2n(2x)^{n-1} = \sum_{n=1}^{\infty} n(2x)^{n-1}, \quad |x| < 1/2.$$

Now we bring back the numerator:

$$\frac{x^2}{(1-2x)^2} = x^2 \sum_{n=1}^{\infty} n(2x)^{n-1} = \sum_{n=1}^{\infty} n2^{n-1}x^{n+1}, \quad |x| < 1/2.$$

Example 4. Use a power series to find

$$\int \frac{\ln(1-t)}{t}\ dt.$$

At first it appears that the integrand has a problem at $t = 0$, but by L'Hopital's Rule,

$$\lim_{t\to 0} \frac{\ln(1-t)}{t} = \lim_{t\to 0} \frac{-1}{1-t} = -1.$$

We focus on the numerator of the integrand first:

$$\ln(1-t) = -\int \frac{1}{1-t}\ dt = -\int \left(\sum_{n=0}^{\infty} t^n\right)\ dt = C - \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}, \quad |t| < 1.$$

We can determine the value of C by evaluating $\ln(1-t)$ at say $t=0$:

$$0 = \ln(1-0) = C - \sum_{n=0}^{\infty} \frac{0^{n+1}}{n+1} = C.$$

Reintroducing the denominator gives the integrand as a power series:

$$\frac{\ln(1-t)}{t} = -\sum_{n=0}^{\infty} \frac{t^n}{n+1}, \quad |t| < 1.$$

Since a convergent power series is continuous on its open interval of convergence, notice that

$$\lim_{t\to 0} \frac{\ln(1-t)}{t} = -\lim_{t\to 0} \sum_{n=0}^{\infty} \frac{t^n}{(n+1)} = -\sum_{n=0}^{\infty} \frac{0^n}{n+1} = -1,$$

in exact agreement with the L'Hopital's Rule calculation done above.

We now integrate term by term:

$$\int \frac{\ln(1-t)}{t}\, dt = C - \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)^2}.$$

We can now use this antiderivative to evaluate definite integrals! That is,

$$\int_{-3/4}^{-1/4} \frac{\ln(1-t)}{t}\, dt = \left[-\sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+2)^2} \right]_{-3/4}^{-1/4} = -\sum_{n=0}^{\infty} \frac{(-1/4)^{n+1}}{(n+1)^2} + \sum_{n=0}^{\infty} \frac{(-3/4)^{n+1}}{(n+1)^2},$$

the difference of two convergent alternating series.