

Math 113 Lecture #34
§11.10: Taylor and Maclaurin Series, Part I

Taylor's Formula for the Coefficients. Last time we saw how to use the geometric series to express certain kinds of functions as power series.

We now learn how to express many functions as power series.

We start by *assuming* that we have a power series with a positive radius of convergence:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad |x-a| < R.$$

We learned last time that such an f is infinitely differentiable.

If we know what $f(x)$ is, how do we find the coefficients c_0, c_1, c_2, \dots ?

Well, if we evaluate the power series at its center, we get

$$f(a) = \sum_{n=0}^{\infty} c_n(a-a)^n = \sum_{n=0}^{\infty} c_n 0^n = c_0 + 0c_1 + 0c_2 + \dots$$

This gives

$$c_0 = f(a).$$

How do we find c_1 ? Well, we evaluate $f'(x)$ at the center:

$$f'(a) = \sum_{n=1}^{\infty} n c_n (a-a)^{n-1} = 1c_1 + 0c_2 + 0c_3 + \dots$$

So we have

$$c_1 = f'(a).$$

It now stands to reason that c_2 is somehow related to the second derivative of f :

$$f''(a) = \sum_{n=2}^{\infty} n(n-1)c_n(a-a)^{n-2} = 1 \cdot 2 \cdot c_2 + 0c_3 + 0c_4 + \dots$$

Thus we have

$$c_2 = \frac{f''(a)}{1 \cdot 2} = \frac{f^{(2)}(a)}{2!}.$$

By $f^{(n)}(a)$ we mean the n^{th} derivative of f evaluated at a .

Continuing the above pattern gives c_3 :

$$f^{(3)}(a) = \sum_{n=3}^{\infty} n(n-1)(n-2)c_n(a-a)^{n-3} = 1 \cdot 2 \cdot 3 \cdot c_3 + 0c_4 + 0c_5 + \dots$$

This gives

$$c_3 = \frac{f^{(3)}(a)}{3!}.$$

By now you should be able to guess the formula for the value of the n^{th} coefficient in the power series for f :

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Putting this into the power series for f gives

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

This power series is called the **Taylor series** of the function f about a .

This power series is called a **Maclaurin series** if $a = 0$.

A word of warning on this Taylor series: we have shown that *if* f can be represented by a convergent power series (i.e., $R > 0$), then f is equal to its Taylor series.

There are infinitely differentiable functions which are not equal to their Taylor series about a , such as

$$f(x) = \begin{cases} \exp(-1/x^2) & x \neq 0, \\ 0 & x = 0, \end{cases}$$

about $a = 0$.

Taylor Polynomials and the Remainder. We now consider the question of whether an infinitely differentiable function f is equal to its Taylor series or not.

This requires an investigation of the sequence of partial sums for the Taylor series and their remainders.

The n^{th} partial sum of a Taylor series for f about a is called the n^{th} -**degree Taylor Polynomial** of f at a :

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k.$$

The limit of the sequence of n^{th} -degree Taylor Polynomials is the sum of the Taylor series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \lim_{n \rightarrow \infty} T_n(x).$$

How good of an approximation $T_n(x)$ is to $f(x)$ is determined by the **remainder**:

$$R_n(x) = f(x) - T_n(x), \text{ i.e., } f(x) = T_n(x) + R_n(x).$$

For $T_n(x)$ to converge to $f(x)$, i.e., for the Taylor series of f to be equal to f , requires that

$$\lim_{n \rightarrow \infty} R_n(x) = 0.$$

All of this discussion is on the interval $|x-a| < R$, where R is the radius of convergence of the Taylor series for the infinitely differentiable function f about a .

Taylor found a way to measure the size of the remainder of a Taylor series.

Taylor's Inequality for the Remainder. If there are constants $M > 0$ and $d > 0$ such that $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \text{ for } |x - a| \leq d.$$

Example 1. We will show that e^x is equal to its Maclaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The function $f(x) = e^x$ is infinitely differentiable, with $f^{(n)}(x) = e^x$, and so

$$f^{(n)}(0) = e^0 = 1.$$

Since $c_n = f^{(n)}(0)/n!$, the Maclaurin series for e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The radius of convergence of this power series is $R = \infty$:

$$\lim_{n \rightarrow \infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < 1.$$

We need to show that the remainder $R_n(x)$ goes to 0 for any x to get that the Maclaurin series for e^x converges to e^x .

For any positive number d we want to find M such that $|f^{(n+1)}(x)| \leq M$ for all $0 \leq x \leq d$.

Since $f^{(n+1)}(x) = e^x$, the maximum value of $f^{(n+1)}(x)$ on $|x| \leq d$ is obtained at the right endpoint:

$$M = e^d.$$

Notice that this upper bound is the same for all orders of derivatives of e^x on $|x| \leq d$.

Applying Taylor's Inequality for the remainder gives

$$|R_n(x)| \leq \frac{e^d}{(n+1)!} |x|^{n+1} \text{ for } |x| \leq d.$$

Since $|x| \leq d$, then

$$|R_n(x)| \leq \frac{e^d d^{n+1}}{(n+1)!} \text{ for } |x| \leq d.$$

The right hand side of this inequality has limit 0 as $n \rightarrow \infty$ for any positive d .

It follows by the Squeeze Theorem that

$$\lim_{n \rightarrow \infty} R_n(x) = 0 \text{ for all } x.$$

Thus e^x is equal to its Maclaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$