# Math 113 Lecture #35
## §11.10: Taylor and Maclaurin Series, Part II

**The Binomial Theorem.** It is easy to expand $(1+x)^k$ when $k$ is an integer, but what if $k$ is any real number?

We will answer this by finding the Maclaurin series for $f(x) = (1+x)^k$.

We compute some of the derivatives of $f$ to discern the pattern in the coefficients of that power series:

$$\begin{array}{lll}
f(x) = (1+x)^k, & f(0) = 1, & c_0 = 1, \\
f'(x) = k(1+x)^{k-1}, & f'(0) = k, & c_1 = k, \\
f''(x) = k(k-1)(1+x)^{k-2}, & f''(0) = k(k-1), & c_2 = \dfrac{k(k-1)}{2!}, \\
f'''(x) = k(k-1)(k-2)(1+x)^{k-3}, & f'''(0) = k(k-1)(k-2), & c_3 = \dfrac{k(k-1)(k-2)}{3!}.
\end{array}$$

Continuing this calculation we identify the pattern for $n \geq 1$:

$$\begin{aligned}
f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1-x)^{k-n}, \\
f^{(n)}(0) &= k(k-1)\cdots(k-n+1), \\
c_n &= \frac{k(k-1)\cdots(k-n+1)}{n!}.
\end{aligned}$$

A standard notation for these coefficient is

$$c_0 = \binom{k}{0} = 1, \; c_n = \binom{k}{n}, n \geq 1.$$

The Maclaurin Series for $(1+x)^k$ is

$$\sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

We compute its radius of convergence by the Ratio Test:

$$\begin{aligned}
&\lim_{n\to\infty} \left| \frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n} \right| \\
&= |x| \lim_{n\to\infty} \left| \frac{k-n}{n+1} \right| = |x|.
\end{aligned}$$

The Maclaurin series for $(1+x)^k$ converges for $|x| < 1$, i.e., it has a radius of convergence $R = 1$.

The Binomial Theorem asserts that the Maclaurin series for $(1+x)^k$ converges to $(1+x)^k$ for any real $k$ and $|x| < 1$:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

When $k$ is a positive integer this becomes

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n \text{ where } \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} = \frac{k!}{n!(k-n)!}.$$

**Example 1.** Use the Binomial Theorem to expand $\sqrt{1+x}$ as a power series.

We use $k = 1/2$ in the Binomial Theorem to get

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{3x^3}{48} - \cdots.$$

Here we understand that

$$\binom{1/2}{0} = 1, \text{ and } \binom{1/2}{1} = \frac{1}{2}, \ \binom{1/2}{2} = \frac{(1/2)(-1/2)}{2} = -\frac{1}{8}, \text{ etc.}$$

## Using a Table of Maclaurin Series.

We can compute the Maclaurin series for several functions and use these to find the power series for other functions.

Here is a Table of some Maclaurin series along with their radii of convergence.

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, & R &= 1,\\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, & R &= \infty,\\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, & R &= \infty,\\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, & R &= \infty,\\
\tan^{-1}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, & R &= 1,\\
(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n, & R &= 1.
\end{aligned}$$

**Example 2.** Find the Maclaurin series for $f(x) = x\sin(\frac{1}{2}x^2)$.

We start with the Maclaurin series for sine:

$$\sin\big((1/2)x^2\big) = \sum_{n=0}^{\infty} \frac{(-1)^n \big((1/2)x^2\big)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{2n+1}(2n+1)!}.$$

Then we multiply this through by $x$:

$$x\sin\big((1/2)x^2\big) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{2^{2n+1}(2n+1)!}.$$

## Using Power Series to Evaluate Limits.

Recall that L'Hospital's Rule is used when we want to resolve indeterminant forms.

Power series expansions can sometimes resolve the indeterminant forms quicker than L'Hospital's Rule.

**Example 3.** Use power series to evaluate

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x}.$$

We use the Maclaurin series for $\cos x$ and $e^x$:

$$\begin{aligned}
\frac{1-\cos x}{1+x-e^x} &= \frac{1-\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}{1+x-\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}} \\
&= \frac{1-\big(1-x^2/2+x^4/24+\cdots\big)}{1+x-\big(1+x+x^2/2+x^3/6+\cdots\big)} \\
&= \frac{x^2/2-x^4/24+\cdots}{-x^2/2-x^3/6+\cdots}.
\end{aligned}$$

From this we readily compute the limit:

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = -1.$$

How many times would we use L'Hospital's Rule to get this?

## Using Power Series to Find Indefinite Integrals.

We have mentioned in the past that there are continuous functions which do not have "nice" antiderivatives.

We can use power series to find these antiderivatives.

**Example 4.** Evaluate

$$\int \frac{\cos x - 1}{x}\, dx.$$

The integrand is defined at $x = 0$ which by L'Hospital's Rule is equal to 0.

We use the Maclaurin series for cosine to "simplify" the integrand:

$$\begin{aligned}
\frac{\cos x - 1}{x} &= \frac{1}{x}\left(-1+\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right) \\
&= \frac{1}{x}\left(-1+1-\frac{x^2}{2}+\frac{x^4}{24}+\cdots\right) \\
&= \frac{1}{x}\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n)!}.
\end{aligned}$$

We now integrate this term-by-term and add the arbitrary constant:

$$\int \frac{\cos x - 1}{x}\, dx = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n(2n)!} + C.$$

## Multiplying and Dividing Power Series.

We can get more power series representations for functions by multiplying and dividing power series.

These operations on power series involve lots of algebra.

There is a Theorem which states that if $f(x) = \sum c_n x^n$ and $g(x) = \sum b_n x^n$ converge for $|x| < R$ where $R > 0$, then $f(x)g(x)$ is a power series convergent for $|x| < R$ as well.

We find the product as if we were multiplying polynomials together.

For division of $f(x)$ by $g(x)$ we require that $b_0 \neq 0$ (i.e., $g(0) \neq 0$), so that $1/g(x)$ is defined for small $|x|$, and by the multiplication Theorem, $f(x)/g(x)$ is a power series convergent for small $|x|$.

**Example 5.** Find the Maclaurin series for $f(x) = e^x \sin x$.

This is given by multiplying the Maclaurin series for $e^x$ and $\sin x$:

$$\begin{aligned}
e^x \sin x &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\\
&= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right)\\
&= 1\left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right)\\
&\quad + x\left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right)\\
&\quad + \frac{x^2}{2}\left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right)\\
&\quad + \frac{x^3}{6}\left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right)\\
&\quad + \frac{x^4}{24}\left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right)\\
&= x + x^2 + \frac{x^3}{3} - \frac{4x^5}{120} + \cdots.
\end{aligned}$$

**Example 6.** Find the Maclaurin series for $f(x) = \sec x$.

We recognize that $\sec x = 1/\cos x$, and so we are asking to divide the Maclaurin series for $\cos x$ into 1.

We divide

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$$

into 1, which we do by long division to get

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots.$$