Math 113 Lecture #36
§11.11: Applications of Taylor Polynomials

The Taylor Polynomial Approximation. Suppose that $f(x)$ is equal to its Taylor series about a with a positive radius of convergence R:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad |x-a| < R.$$

Then the sequence of n^{th}-degree Taylor polynomials

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

converges to $f(x)$ on $|x-a| < R$ as $n \to \infty$.

This means that each T_n is an approximation of $f(x)$ near $x = a$, with the error of the approximation measured by the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|.$$

The first-degree Taylor polynomial is nothing more than the tangent line approximation:

$$T_1(x) = f(a) + f'(a)(x-a) \approx f(x) \text{ near } x = a.$$

How good this linear approximation is near $x = a$ is determined by Taylor's Inequality: if $|f''(x)| \leq M$ for $|x-a| \leq d$, then

$$|R_1(x)| \leq \frac{M}{2}|x-a|^2 \text{ for } |x-a| \leq d.$$

The second-degree Taylor polynomial is called the quadratic approximation:

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \approx f(x) \text{ near } x = a.$$

How good this quadratic approximation is near $x = a$ is determined by Taylor's Inequality: if $|f'''(x)| \leq M$ for $|x-a| \leq d$, then

$$|R_2(x)| \leq \frac{M}{6}|x-a|^3 \text{ for } |x-a| \leq d.$$

We can determine the error of the n^{th}-degree Taylor polynomial approximation in a similar way for $n \geq 3$.

If the Taylor series is an alternating series, we can use the Alternating Series Estimate to determine the error of the Taylor polynomials.

And if worse comes to worse, we can simply plot the graph of $R_n(x)$ to see what the error of the approximation is.

Example 1. Approximate $f(x) = \ln(1+2x)$ with the third-degree Taylor Polynomial about $x = 1$, and determine the error of this approximation on the interval $1/2 \le x < 3/2$.

There is no need to find all of the coefficients in the Taylor series for $f(x)$, since we are only interested in T_3:

$f(x) = \ln(1+2x), \quad f'(x) = 2(1+2x)^{-1}, \quad f''(x) = -4(1+2x)^{-2}, \quad f'''(x) = 16(1+2x)^{-3}.$

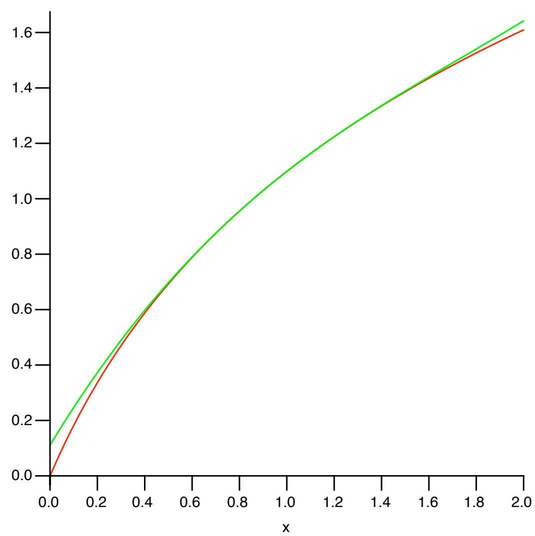
We compute the first four coefficients in the Taylor series for $f(x)$:

$$c_0 = \ln 3, \quad c_1 = \frac{2}{3}, \quad c_2 = -\frac{4}{3^2 2} = -\frac{2}{9}, \quad c_3 = \frac{16}{3^3 3!} = \frac{8}{81}.$$

The third-degree Taylor polynomial for $f(x)$ about $a = 1$ is

$$T_3(x) = \ln 3 + \frac{2(x-1)}{3} - \frac{2(x-1)^2}{9} + \frac{8(x-1)^3}{81}.$$

Here are the graphs of $f(x) = \ln(1+2x)$ and $T_3(x)$.



The green (or upper) graph is $T_3(x)$ and the red (or lower) is $f(x)$: how can we tell?

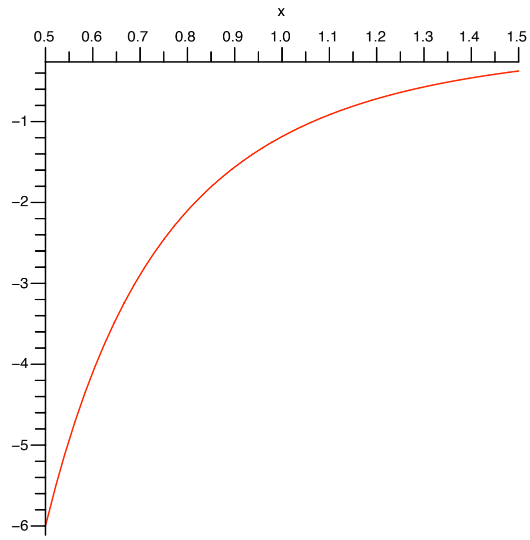
From the graph it appears that $T_3(x)$ is a good approximation to $f(x)$ on the interval $[1/2, 3/2]$.

We use Taylor's Inequality to say how good, for which we need the fourth derivative:

$$f^{(4)}(x) = -96(1+2x)^{-4}.$$

We find an M for which $|f^{(4)}(x)| \le M$ on $|x-1| \le 1/2$ by Calculus (i.e., find the maximum value of the function on that interval), or a slightly rougher M by graphing the fourth derivative.

Here is the graph of $f^{(4)}(x)$ on $1/2 \le x \le 3/2$, i.e., on $|x-1| \le 1/2$.



From the graph we see that $M = 6$, and so the error of $T_3(x)$ as an approximation of $f(x)$ on $|x-1| \leq 1/2$ is

$$|R_3(x)| \leq \frac{6}{4!}|x-1|^4 = \frac{(x-1)^4}{4} \leq \frac{1}{2^4 4} = \frac{1}{64} \approx 0.016.$$

What degree n of a Taylor polynomial $T_n(x)$ would we need to get $|R_n(1.4)| \leq 0.005$?

Evaluating the estimate for $R_3(x)$ when $x = 1.4$ we get

$$|R_3(1.4)| \leq \frac{|1.4-1|^4}{4} = 0.0064.$$

This is not small enough, so we consider $T_4(x) = T_3(x) + c_4 x^4$, where, since $f^{(4)}(x) = -96(1+2x)^{-4}$, we have

$$c_4 = \frac{f^{(4)}(1)}{4!} = \frac{-96}{3^4 4!} = \frac{-4}{3^4} = -\frac{4}{81}.$$

For Taylor's Inequality for T_4 we need the maximum of the absolute value of $f^{(5)}(x) = 768(1+2x)^{-5}$ on $|x-1| \leq 1/2$, which, because $f^{(5)}$ is decreasing, occurs at the left-hand endpoint:

$$M = f^{(5)}(1/2) = \frac{768}{(1+1)^5} = 24.$$

Thus on the interval $|x-1| \leq 1/2$, we have

$$|R_4(x)| \leq \frac{M}{(4+1)!}|x-1|^5 \leq \frac{24}{120}(1/2)^5 = 0.00625.$$

More specifically we have

$$|R_4(1.4)| \leq \frac{24}{120}|1.4-1|^5 = 0.002048 \leq 0.005.$$