11.5 Consequences of Cauchy's Integral Formula

March 17, 2020

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Some results hold only for holomorphic functions of the form

$$f : U \to \mathbb{C}.$$

We will be sure to indicate which it is in the results.

Definition. For U a subset of $\mathbb{C}$, a function $f : U \to X$ is bounded if there exists $M > 0$ such that

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Theorem 11.5.1 (Liouville's Theorem). If $f : \mathbb{C} \to X$ is

- entire, and
- bounded,

then

- f is a constant function.

Example 11.5.2. The entire functions $\cos(z)$ and $\sin(z)$ are bounded but not constant when $z \in \mathbb{R}$.

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You are given a hint for $\sin(z)$ in Exercise 11.20, but here are some better hints: for $z = x + iy$, there holds

$$\sin(z) = \sin x \cosh y + i \cos x \sinh y,$$

$$\cos(z) = \cos x \cosh y - i \sin x \sinh y.$$

$\cosh y = \frac{e^y + e^{-y}}{2}$ $\sinh y = \frac{e^y - e^{-y}}{2}$

What questions do you have?

Example. A complex Banach space is the complex vector space $M_n(\mathbb{C})$ equipped with the induced matrix norm $\|\cdot\|_\infty$. The function $f : \mathbb{C} \to M_2(\mathbb{C})$ defined by

$$f(z) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}$$

is entire because for any $z_0 \in \mathbb{C}$ we have

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{1}{z - z_0} \begin{bmatrix} 0 & 0 \\ 0 & z - z_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

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The entire function f is not constant, and so by the contrapositive of Liouville's Theorem its norm is not bounded; explicitly we have

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as $|z| \to \infty$.
This matrix valued function f is readily generalized to $n \geq 3$.

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Corollary 11.5.3. If $f : \mathbb{C} \to \mathbb{C}$ is entire and f is uniformly bounded away from zero, then f is constant.

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Connection with Liouville's Theorem and restriction to complex-valued entire functions:

$$\left|\frac{1}{f(z)}\right| \leq \frac{1}{\epsilon}.$$

(Do not have multiplicative inverses for all nonzero elements in general Banach spaces; that is why we restrict to complex-valued functions here.)

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The lengthy proof is by contradiction.

Remark. The Fundamental Theorem of Algebra is an existence result – its proof does not give an algorithm for finding the roots. You have it as HW (Exercise 11.21) to show that a polynomial $p_n(z)$ of degree n has exactly n roots (counting multiple roots). Hint: use the Fundamental Theorem of Algebra to find a root, say z_n of $p_n(z)$, then form a new polynomial $p_{n-1}(z)$ of degree $n-1$ obtained by dividing $p_n(z)$ by the factor $z - z_n$. Is there a root z_{n-1} of p_{n-1}?

What questions do you have?

Overview of Maximum Modulus Principle

For an open set U in $\mathbb{C}$ and a holomorphic function $f : U \to \mathbb{C}$, the continuous function $z \to |f(z)|$, on any compact subset K of U, attains its maximum value at some point of K by the Extreme Value Theorem.

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The Maximum Modulus Principle is a consequence of the following two Lemmas that apply to general complex Banach spaced valued holomorphic functions.

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The Maximum Modulus Principle is a consequence of the following two Lemmas that apply to general complex Banach spaced valued holomorphic functions.

The book states Lemma 11.5.6 only for complex-valued holomorphic functions, but its proof works for general complex Banach spaced value holomorphic functions, and we present this way.

Lemma 11.5.6. For an open U in $\mathbb{C}$ and $f : U \to X$ holomorphic, if

- $\|f\|_X$ attains its supremum at $z_0 \in U$,

then

- $\|f\|_X$ is constant in every open ball $B(z_0, r)$ whose closure $\overline{B(z_0, r)}$ is contained in U.

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Lemma (Precursor to Maximum Modulus Theorem). For U an open, path-connected subset of $\mathbb{C}$ and $f : U \to X$ holomorphic, if

- $\|f\|_X$ is not constant on U,

then

- the continuous function $z \to \|f(z)\|_X$ never attains its supremum on U.

What questions do you have?

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For the open, path-connected set $U = B(0,2)$ in $\mathbb{C}$, the restriction $f : U \to M_2(\mathbb{C})$ is holomorphic and in terms of the induced matrix norm $\| \cdot \|_\infty$, we have

$$\|f(z)\|_\infty = \begin{cases} 1 & \text{if } z \in \overline{B(0,1)}, \\ |z| & \text{if } z \in B(0,2) \setminus \overline{B(0,1)}. \end{cases}$$

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The nonconstant function $\|f\|_\infty$ does not attain is supremum of 2 on U.

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For the open, path-connected set $U = B(0,1)$ in $\mathbb{C}$, the restriction $f : U \to M_2(\mathbb{C})$ is a nonconstant holomorphic function.
But in terms of the induced matrix norm $\|\cdot\|_\infty$, we have $\|f(z)\|_\infty = 1$ for all $z \in U$.

Theorem 11.5.5 (The Maximum Modulus Principle). For an open, path-connected U in $\mathbb{C}$ and a holomorphic $f : U \to \mathbb{C}$, if

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The next result, a corollary of the Maximum Modulus Principle, is stated in an imprecise manner in the book. Here is a precise version.

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Corollary 11.5.7. For a compact set D whose interior D° is nonempty and path-connected, if

- $f : D \to \mathbb{C}$ is continuous and holomorphic on D°,

then

- $|f|$ attains is maximum on ∂D. boundary of D

What questions do you have?