12.3 The Resolvent

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The Jordan Canonical Form, or spectral decomposition, of a linear operator on a finite dimension vector space has important applications in many areas such as differential equations and dynamical systems (stability and control theory).

Computing the Jordan Canonical form depends on finding bases for the generalized eigenspaces, and as mentioned before, this is a poorly conditioned numerical algorithm.

A more powerful approach to finding the spectral decomposition of a linear operator on a finite dimensional vector space uses the tools of complex analysis.

This theoretical approach is basis-free, meaning we do not have to find bases of the generalized eigenspaces to get the spectral decomposition.

Definition 12.3.1. The resolvent set of $A \in M_n(\mathbb{C})$, denoted by $\rho(A)$, is the set of points $z \in \mathbb{C}$ for which $zI - A$ is invertible.

The complement

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$

is the spectrum of A.

The resolvent of A is the function $R(A, \cdot) : \rho(A) \to M_n(\mathbb{C})$ defined by

$$R(A, z) = (zI - A)^{-1}.$$

We will sometimes make use of the notation $R_A(z)$ for $R(A, z)$ to make clear the dependence of the resolvent on A.

When there is no ambiguity, we denote $R(A, z)$ simply as $R(z)$.

First Reading Quiz Question. What kind of function is the resolvent of a linear operator?

Example (in lieu of 12.3.2). The resolvent R_A for

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \in M_2(\mathbb{C})$$

is $R_A(z) = (zI - A)^{-1}$, i.e.,

$$\begin{aligned} R_A(z) &= \begin{bmatrix} z-1 & -1 \\ -4 & z-1 \end{bmatrix}^{-1} \\ &= \frac{1}{(z-1)^2 - 4} \begin{bmatrix} z-1 & 1 \\ 4 & z-1 \end{bmatrix} \\ &= \frac{1}{z^2 - 2z - 3} \begin{bmatrix} z-1 & 1 \\ 4 & z-1 \end{bmatrix} \\ &= \frac{1}{(z-3)(z+1)} \begin{bmatrix} z-1 & 1 \\ 4 & z-1 \end{bmatrix} \end{aligned}$$

The resolvent has simple poles precisely on $\sigma(A) = \{3, -1\}$.

The domain of the R_A is the resolvent set $\rho(A) = \mathbb{C} \setminus \sigma(A)$.

Remark 12.3.3. Each entry of the resolvent is a rational function by Cramer's Rule:

$$R_A(z) = (zI - A)^{-1} = \frac{1}{\det(zI - A)}\operatorname{adj}(zI - A)$$

where $\det(zI - A)$ is the characteristic polynomial of A, and

$$\operatorname{adj}(zI - A)$$

is the adjugate matrix, i.e., the transpose of the matrix of signed minors of $zI - A$, which satisfies

$$\boxed{(zI - A)\operatorname{adj}(zI - A) = \det(zI - A)I}$$

(see Definition 2.9.19 and Theorem 2.9.22).

The rational function nature of the entries of the resolvent shows that the resolvent has poles, some possibly not simple, precisely on $\sigma(A)$.

What questions do you have?

monic polynomial

= leading coefficient is 1

Example (in lieu of 12.3.4). (i) Find the resolvent for

$$A = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix}.$$

Because A is upper triangular, the characteristic polynomial of A is

$$\det(zI - A) = (z-6)^2(z-4).$$

monic

$z^3 + ?z^2 + ?z + ?$

The adjugate of A is

$$\operatorname{adj}(zI - A) = \begin{bmatrix} (z-6)(z-4) & z-4 & 7 \\ 0 & (z-6)(z-4) & 7(z-6) \\ 0 & 0 & (z-6)^2 \end{bmatrix}.$$

We verify this by computing $(zI - A)\operatorname{adj}(zI-A) = \det(zI-A)I$. ✓

The resolvent of

$$A = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix}$$

is

$$R(z) = \begin{bmatrix} (z-6)^{-1} & (z-6)^{-2} & 7(z-6)^{-2}(z-4)^{-1} \\ 0 & (z-6)^{-1} & 7(z-6)^{-1}(z-4)^{-1} \\ 0 & 0 & (z-4)^{-1} \end{bmatrix}.$$

Each nonzero entry has at least one isolated singularity occurring at one of the eigenvalues of A.

Some of the isolated singularities are readily identified as simple poles, others as poles of order 2, while for the remaining ones the method of partial fractions identifies them.

(ii) Because

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

is upper triangular, the characteristic polynomial of A is

$$\det(zI - A) = (z-2)^3(z-5).$$

The matrix $\mathrm{adj}(zI - A)$ is

$$\begin{bmatrix} (z-2)^2(z-5) & (z-2)(z-5) & z-5 & 3 \\ 0 & (z-2)^2(z-5) & (z-2)(z-5) & 3(z-2) \\ 0 & 0 & (z-2)^2(z-5) & 3(z-2)^2 \\ 0 & 0 & 0 & (z-2)^3 \end{bmatrix}.$$

The resolvent is

$$R_A(z) = \frac{1}{\det(zI - A)} \mathrm{adj}(zI - A).$$

What questions do you have?

Lemma 12.3.5. Let $A, A_1, A_2 \in M_n(\mathbb{C})$.

(i) If $z_1, z_2 \in \rho(A)$, then

$$R(z_2) - R(z_1) = (z_1 - z_2)R(z_2)R(z_1).$$

[This is known as Hilbert's Identity.]

(ii) If $z \in \rho(A_1) \cap \rho(A_2)$, then

$$R(A_2, z) - R(A_1, z) = R(A_1, z)(A_2 - A_1)R(A_1, z).$$

(iii) If $z \in \rho(A)$, then

$$R(z)A = AR(z).$$

(iv) If $z_1, z_2 \in \rho(A)$, then

$$R(z_1)R(z_2) = R(z_2)R(z_1).$$

We show that the resolvent R_A is a matrix-valued holomorphic function on $\rho(A)$ by finding power series expansions of R_A at all points $z \in \rho(A)$.

Let $\|\cdot\|$ be a matrix norm on $M_n(\mathbb{C})$, i.e., a norm on $M_n(\mathbb{C})$ that for all $A, B \in M_n(\mathbb{C})$ satisfies

$$\boxed{\|AB\| \le \|A\|\,\|B\|.}$$

Examples of matrix norms are the induced p-norms $\|\cdot\|_p$ and the Frobenius norm $\|\cdot\|_F$.

Theorem 12.3.6. For $A \in M_n(\mathbb{C})$, the resolvent set $\rho(A)$ is open, and R is holomorphic on $\rho(A)$ where for each $z_0 \in \rho(A)$, the resolvent is given by the power series

$$R(z) = \sum_{k=0}^{\infty} (-1)^k (z - z_0)^k R^{k+1}(z_0),$$

whose the radius of convergence is at least as large as $\|R(z_0)\|^{-1}$.

Remark 12.3.7. Comparing the power series

$$R(z) = \sum_{k=0}^{\infty} (-1)^k (z - z_0)^k R^{k+1}(z_0)$$

Neumann Series

in Theorem 12.3.6 and the Taylor series

$$R(z) = \sum_{k=0}^{\infty} \frac{R^{(k)}(z_0)}{k!} (z - z_0)^k$$

reveals, by the uniqueness of the Taylor series, a relationship between the powers of R and its derivatives, namely that

$$\boxed{(-1)^k R^{k+1}(z_0) = \frac{R^{(k)}(z_0)}{k!}.}$$

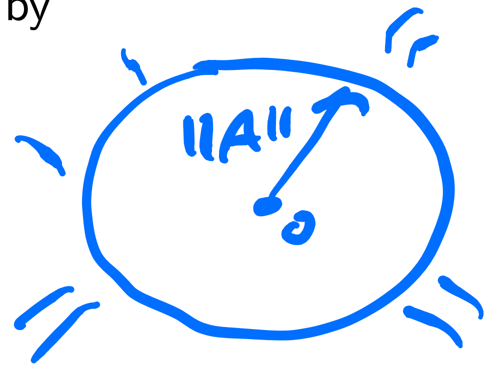
As this holds for every $z_0 \in \rho(A)$, we may replace z_0 by z to get

$$R^{(k)}(z) = k!(-1)^k R^{k+1}(z) = k!(-1)^k (zI - A)^{-(k+1)}.$$

What questions do you have?

Theorem 12.3.8. For $A \in M_n(\mathbb{C})$, the Laurent series of $R(z)$ on the open annulus $|z| > \|A\|$ exists and is given by

$$R(z) = \sum_{k=0}^{\infty} \frac{A^k}{z^{k+1}}.$$



Proof. The resolvent is

$$R(z) = (zI - A)^{-1} = (z(I - z^{-1}A))^{-1} = z^{-1}(I - z^{-1}A)^{-1}.$$

To express $(I - z^{-1}A)^{-1}$ as a Neumann series requires that $\|z^{-1}A\| < 1$.

This condition gives the annulus $|z| > \|A\|$, on which

$$R(z) = z^{-1}(I - z^{-1}A)^{-1} = z^{-1}\sum_{k=0}^{\infty}\frac{A^k}{z^k} = \sum_{k=0}^{\infty}\frac{A^k}{z^{k+1}}$$

$$= \frac{I}{z} + \frac{A}{z^2} + \frac{A^2}{z^3} + \cdots .$$

Remark. One might be tempted to say that the resolvent $R(z)$ has an essential singularity at $z = 0$, but the open annulus $|z| > \|A\|$ is not a punctured disk unless $A = 0$, in which case the resolvent has a simple pole at 0.

But we can use the Laurent series of $R(z)$ on $|z| > \|A\|$ to say something about the behavior of $R(z)$ as $|z| \to \infty$.

For this we speak of $R(z)$ being holomorphic in a neighbourhood of ∞, which means, for the change of variables $z = g(w) = 1/w$, that the function $(R \circ g)(w)$ is holomorphic on an open disk centered at $w = 0$.

Here $|z| = 1/|w|$ so that $|w| \to 0$ if and only if $|z| \to \infty$.

Corollary 12.3.9. For any $A \in M_n(\mathbb{C})$, the resolvent R is holomorphic in a neighbourhood of ∞, and moreover there holds

$$\lim_{|z| \to \infty} \|R(z)\| = 0.$$

Remark 12.3.11. The values of z for which $R_A(z) = (zI - A)^{-1}$ is not defined are precisely the eigenvalues z of A.

Here is another proof that $\sigma(A) \neq \emptyset$.

Corollary 12.3.12. For any $A \in M_n(\mathbb{C})$, the spectrum $\sigma(A)$ is not empty.

Proof. Suppose to the contrary that $\sigma(A) = \emptyset$.

Then $\rho(A) = \mathbb{C}$, and hence the holomorphic resolvent R is entire.

Corollary 12.3.9 implies that $\|R\|$ is a bounded entire function.

Liouville's Theorem implies that R is a constant function.

Since $\lim_{|z|\to\infty} \|R(z)\| = 0$, the resolvent R is the zero function.

But this is a contradiction because the resolvent satisfies

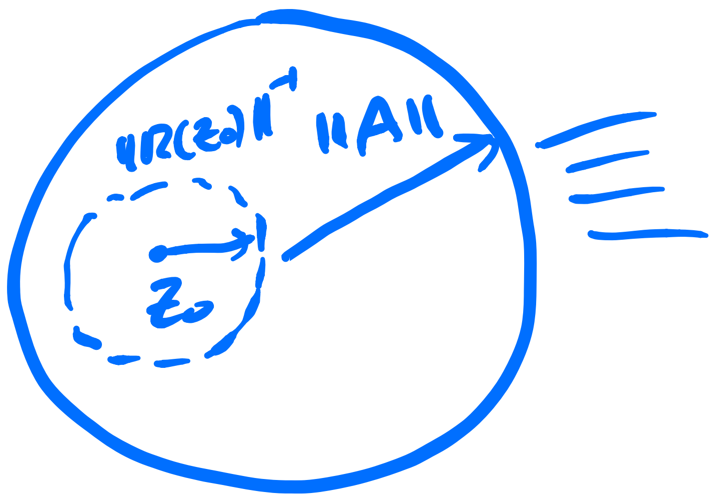
$$I = (zI - A)R(z). = 0$$

What questions do you have?

Note. By Theorem 12.3.6, the radius of convergence of the power series of $R(z)$ about $z_0 \in \rho(A)$ is *at least as large as* $\|R(z_0)\|^{-1}$.

By Theorem 12.3.8, the inner radius of the annulus for the Laurent series of $R(z)$ about 0 is *at least as small as* $\|A\|$.

[Draw a picture.]



We show that these radii can be improved through a limit quantity known as the *spectral radius*.

Lemma 12.3.13. For any $A \in M_n(\mathbb{C})$, the limit

$$\boxed{r(A) = \lim_{k\to\infty} \|A^k\|^{1/k}}$$

exists and is bounded above by $\|A\|$.

Part of proof. We show the "bounded above" part, assuming the limit exists.

Since $\|\cdot\|$ is a matrix norm, we have

$$\|A^k\| \leq \|A\|^k.$$

Thus

$$\|A^k\|^{1/k} \leq \|A\|^{k/k} = \|A\|.$$

Hence

$$r(A) = \lim_{k\to\infty} \|A^k\|^{1/k} \leq \lim_{k\to\infty} \|A\| = \|A\|.$$

Remark. We will see in the next section a result stating that the quantity $r(A)$ is independent of the matrix norm used in the limit. An example later today will illustrate this.

Definition. The spectral radius of $A \in M_n(\mathbb{C})$ is defined to be the quantity $r(A)$.

Theorem 12.3.14. For $A \in M_n(\mathbb{C})$, the power series

$$R(z) = \sum_{k=0}^{\infty} (-1)^k (z - z_0)^k R^{k+1}(z_0)$$

converges on $|z - z_0| < [r(R(z_0))]^{-1}$, and the Laurent series

$$R(z) = \sum_{k=0}^{\infty} \frac{A^k}{z^{k+1}}$$

converges on the annulus $|z| > r(A)$.

Second Reading Quiz Question: The resolvent $R_A(z)$ has a Laurent series on $|z| > r(A)$. True.

Remark. The radii of convergence given in Theorem 12.3.14 do improve the radii $\|R(z_0)\|^{-1}$ and $\|A\|$ given in Theorem 12.3.6 and Theorem 12.3.8.

For the first radii this is because $r(R(z_0)) \leq \|R(z_0)\|$ implies

$$\|R(z_0)\|^{-1} \leq [r(R(z_0))]^{-1}.$$

For the second radii this is because

$$r(A) \leq \|A\|,$$

i.e., a smaller inner radius of the open annulus is possible.

Remark 12.3.15. A lower bound on the spectral radius $r(A)$ is given by quantity

$$\sigma_M = \sup\{|\lambda| : \lambda \in \sigma(A)\}.$$

Justification of this lower bound follows from Theorem 12.3.14 because the open annulus on which the Laurent series converges cannot include any point in $\sigma(A)$ where the resolvent has poles.

What questions do you have?

Example. Find the spectral radius of

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \in M_2(\mathbb{C})$$

with respect to the 1-norm, the ∞-norm, and the Frobenius norm.

The matrix A is simple since it has two distinct eigenvalues 3 and -1, hence A is diagonalizable.

Eigenvectors corresponding to $\lambda_1 = 3$ and $\lambda_2 = -1$ are

$$\xi_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \xi_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The matrix

$$P = \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

diagonalizes A, i.e.,

$$\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = D = P^{-1}AP = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}.$$

Rewritten we have

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix},$$

so that

$$\begin{aligned} A^k &= \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix} \\ &= \begin{bmatrix} 3^k & (-1)^k \\ 2(3^k) & -2(-1)^k \end{bmatrix} \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix} \\ &= \begin{bmatrix} (1/2)(3^k) + (1/2)(-1)^k & (1/4)(3^k) - (1/4)(-1)^k \\ 3^k - (-1)^k & (1/2)(3^k) + (1/2)(-1)^k \end{bmatrix}. \end{aligned}$$

In terms of the 1-norm, the first column gives the maximum:

$$\|A^k\|_1 = \frac{3}{2}(3^k) - \frac{1}{2}(-1)^k.$$

In terms of the ∞-norm, the second row gives the maximum:

$$\|A^k\|_\infty = \frac{3}{2}(3^k) - \frac{1}{2}(-1)^k.$$

Since $\|A^k\|_1 = \|A^k\|_\infty$ there will hold

$$\lim_{k\to\infty} \|A^k\|_1^{1/k} = \lim_{k\to\infty} \|A^k\|_\infty^{1/k}.$$

We proceed with computing the spectral radius for the 1-norm:

$$r(A) = \lim_{k\to\infty} \|A^k\|_1^{1/k} = \lim_{k\to\infty} \left[\frac{3}{2}(3^k) - \frac{1}{2}(-1)^k\right]^{1/k}.$$

[For HW Exercise 12.14 you will use the $\exp \circ \log$ and L'Hospital's Rule approach to get the exact limit.]

The term $-(1/2)(-1)^k$ is bounded in absolute value by $1/2$; heuristically this term makes no contribution to the limit; hence,

$$r(A) = \lim_{k\to\infty} \left[\frac{3}{2}(3)^k\right]^{1/k} = \lim_{k\to\infty} \left(\frac{3}{2}\right)^{1/k} (3) = 3.$$

The spectral radius for the ∞-norm is also 3, the largest moduli of the eigenvalues.

With

$$A^k = \begin{bmatrix} (1/2)(3^k) + (1/2)(-1)^k & (1/4)(3^k) - (1/4)(-1)^k \\ 3^k - (-1)^k & (1/2)(3^k) + (1/2)(-1)^k \end{bmatrix}$$

the Frobenius norm of A^k is

$$\begin{aligned} \|A_k\|_F^2 &= \underline{\mathrm{tr}((A_k)^{\mathrm{H}} A_k)} \\ &= 2\left(\frac{1}{2}(3^k) + \frac{1}{2}(-1)^k\right)^2 + ((3^k) - (-1)^k)^2 \\ &\quad + \left(\frac{1}{4}(3^k) - \frac{1}{4}(-1)^k\right)^2 \\ &= \frac{1}{2}(3^{2k}) + (3^{2k}) + \frac{1}{16}(3^{2k}) \cdots \\ &= \frac{25}{16}(3^{2k}) = \underline{\left(\frac{5}{4}\right)^2 (3^{2k})} \cdots . \end{aligned}$$

Heuristically we obtain

$$\|A^k\|_F^{1/k} \approx \left(\frac{5}{4}\right)^{1/k} (3) \to \underline{3} = r(A).$$

What questions do you have?