12.5 Spectral Decomposition I

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For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, there exist $A_k \in M_n(\mathbb{C})$, $k \in \mathbb{Z}$, (depending on λ) such that the resolvent of A as a Laurent series about λ has the form

$$R_A(z) = \sum_{k=-\infty}^{\infty} A_k (z-\lambda)^k.$$

By the Laurent Expansion Theorem, we have for each $k \in \mathbb{Z}$ that

$$A_k = \frac{1}{2\pi i} \oint_\Gamma \frac{R_A(z)}{(z-\lambda)^{k+1}} \, dz$$

for a positively oriented simple closed contour Γ enclosing λ but no other element of $\sigma(A)$.

The coefficient $A_{-1} = \mathrm{Res}(R_A(z), \lambda)$ is the spectral projection P_λ.

We are going to discover the nature of the relationships that exist among all the coefficient matrices A_k in Laurent series for $R_A(z)$ about λ.

Nota Bene 12.5.1. Be aware that A_k is a coefficient matrix in a Laurent series

$$R_A(z) = \sum_{k=-\infty}^{\infty} A_k(z-\lambda)^k,$$

while A^k is the k^{th} power of A.

(Horrible) Notation. For $n \in \mathbb{Z}$, define

$$\eta_n = \begin{cases} 1 & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

This is a characteristic or indicator function on the set $\mathbb{Z}$.

Lemma 12.5.2. For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, let Γ and Γ' be two positively oriented simple closed contours in $\rho(A)$ enclosing λ and no other element of $\sigma(A)$. Assume further that Γ is in the interior of Γ', that $z' \in \Gamma'$, and $z \in \Gamma$. Then for all $m \in \mathbb{N}$ there holds

$$\text{(i)}\ \frac{1}{2\pi i} \oint_{\Gamma} (z-\lambda)^{-m-1}(z'-z)^{-1}\ dz = \eta_m (z'-\lambda)^{-m-1},$$

and for all $n \in \mathbb{N}$ there holds

$$\text{(ii)}\ \frac{1}{2\pi i} \oint_{\Gamma'} (z'-\lambda)^{-n-1}(z'-z)^{-1}\ dz' = (1-\eta_n)(z-\lambda)^{-n-1}.$$

[Draw the picture.]

What questions do you have?

Lemma 12.5.3. The matrix coefficients A_k in the Laurent expansion

$$R_A(z) = \sum_{k=-\infty}^{\infty} A_k(z-\lambda)^k$$

about $\lambda \in \sigma(A)$ satisfy

$$\boxed{A_m A_n = (1-\eta_m-\eta_n)A_{m+n+1}}.$$

Remark 12.5.4. Since $P_\lambda = A_{-1}$, Lemma 12.5.3 gives another proof that

$$P_\lambda^2 = A_{-1}A_{-1} = (1-\eta_{-1}-\eta_{-1})A_{-1-1+1} = A_{-1} = P_\lambda.$$

Notation. To express the relationships that exists among the coefficient matrices A_k in the Laurent series of $R_A(z)$ about λ, we define

$$D_\lambda = A_{-2} \text{ and } S_\lambda = A_0.$$

Lemma 12.5.5. For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, there holds

(i) $A_{-n} = D_\lambda^{n-1}$ for all $n \geq 2$,

(ii) $A_n = (-1)^n S_\lambda^{n+1}$ for all $n \geq 1$,

(iii) the spectral projection P_λ commutes with D_λ and with S_λ, where in particular,

$$P_\lambda D_\lambda = D_\lambda,\ P_\lambda S_\lambda = 0,$$

(iv) The Laurent series of $R_A(z)$ about λ is

$$R_A(z) = \frac{P_\lambda}{z-\lambda} + \sum_{k=1}^{\infty} \frac{D_\lambda^k}{(z-\lambda)^{k+1}} + \sum_{k=0}^{\infty} (-1)^k (z-\lambda)^k S_\lambda^{k+1},$$

(v) the spectral projection P_λ commutes with $R_A(z)$, where in particular

$$P_\lambda R_A(z) = \frac{P_\lambda}{z-\lambda} + \sum_{k=1}^{\infty} \frac{D_\lambda^k}{(z-\lambda)^{k+1}}.$$

The proof of these is HW (Exercises 12.23, 12.24, and 12.25).

Remark. The Laurent series for $R_A(z)$ about $\lambda \in \sigma(A)$ is completely determined by the three matrices

$$P_\lambda = A_{-1},\ D_\lambda = A_{-2},\ \text{and}\ S_\lambda = A_0,$$

according to Lemma 12.5.5 part (iv):

$$R_A(z) = \frac{P_\lambda}{z-\lambda} + \sum_{k=1}^{\infty} \frac{D_\lambda^k}{(z-\lambda)^{k+1}} + \sum_{k=0}^{\infty} (-1)^k (z-\lambda)^k S_\lambda^{k+1}.$$

Second Reading Quiz Question: The Laurent series of $R_A(z)$ about $\lambda \in \sigma(A)$ is determined by three matrices?

True

First Reading Quiz Question: How it is possible to write the Laurent series for the resolvent of A about an eigenvalue λ of A using only the spectral projections and D_λ?

What questions do you have?

A Long Example. We verify some parts of Lemma 12.5.5 for the linear operator

$$A = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix}$$

and use other parts of Lemma 12.5.5 to compute Laurent series expansion of $R_A(z)$ about $\lambda = 6$.

We computed previously that $R_A(z)$ is

$$\frac{1}{z-6}\begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{(z-6)^2}\begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{z-4}\begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix},$$

so that the spectral projections are

$$P_6 = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } P_4 = \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Also from the partial fraction decomposition of $R_A(z)$ we have

$$D_6 = \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } D_4 = 0,$$

the latter since there is no $1/(z-4)^2$ term in the partial fraction decomposition.

We may thus neatly write the partial fraction decomposition of the resolvent of A as

$$R_A(z) = \frac{P_6}{z-6} + \frac{D_6}{(z-6)^2} + \frac{P_4}{z-4},$$

where

$$P_6 = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } P_4 = \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Part (iii) of Lemma 12.5.5 states that P_λ commutes with D_λ and S_λ, and that $P_\lambda D_\lambda = D_\lambda P_\lambda = D_\lambda$ and $P_\lambda S_\lambda = S_\lambda P_\lambda = 0$.

Verifying a piece of part (iii) of Lemma 12.5.5, the matrices P_6 and D_6 satsify

$$\begin{aligned} P_6 D_6 &= \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = D_6 \\ &= \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} = D_6 P_6. \end{aligned}$$

The matrix D_6 satisfies

$$D_6^2 = \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Thus $D_6^k = 0$ for all $k \geq 2$, so that the principle part of the Laurent series satisfies

$$\frac{P_6}{z-6} + \sum_{k=1}^{\infty} \frac{D_6^k}{(z-6)^{k+1}} = \frac{P_6}{z-6} + \frac{D_6}{(z-6)^2}.$$

The resolvent $R_A(z)$ has a pole of order 2 are the isolated singularity $\lambda = 6$.

We could compute S_6 by writing $1/(z-4)$ in

$$R_A(z) = \frac{P_6}{z-6} + \frac{D_6}{(z-6)^2} + \frac{P_4}{z-4}$$

as a geometric series in $z-6$, but we won't this time.

What questions do you have?

We will proceed with the long example.

Instead we make use of parts (iv) and (v) of Lemma 12.5.5 to find S_6 (and use the geometric series to verify the work).

First, by part (iv) we have

$$\sum_{k=0}^{\infty}(-1)^k(z-6)^k S_6^{k+1} = R_A(z) - \left(\frac{P_6}{z-6} + \frac{D_6}{(z-6)^2}\right).$$

By part (v) we have

$$R_A(z)P_6 = \frac{P_6}{z-6} + \frac{D_6}{(z-6)^2}.$$

Combining these gives

$$\sum_{k=0}^{\infty}(-1)^k(z-6)^k S_6^{k+1} = R_A(z) - R_A(z)P_6 = R_A(z)(I-P_6) = R_A(z)P_4$$

by the completeness $P_6 + P_4 = I$.

We will determine what $R_A(z)P_4$ is.

Recalling that

$$R_A(z) = \frac{P_6}{z-6} + \frac{D_6}{(z-6)^2} + \frac{P_4}{z-4}$$

we have in the product

$$R_A(z)P_4 = \frac{P_6P_4}{z-6} + \frac{D_6P_4}{(z-6)^2} + \frac{P_4^2}{z-4}$$

that $P_6P_4 = 0$ and $P_4^2 = P_4$, but what is D_6P_4?

It is

$$\begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} = 0.$$

Is this just a coincidence?

With $D_4 = 0$, it is not according to part (v) of Lemma 12.5.5 which gives

$$R_A(z)P_4 = \frac{P_4}{z-4} + \sum_{k=1}^{\infty} \frac{D_4^k}{(z-\lambda)^{k+1}} = \frac{P_4}{z-4},$$

and in comparison with

$$R_A(z)P_4 = \frac{P_6 P_4}{z-6} + \frac{D_6 P_4}{(z-6)^2} + \frac{P_4}{z-4}$$

implies that $D_6 P_4 = 0$.

The point of all of this is that we have

$$\sum_{k=0}^{\infty} (-1)^k (z-6)^k S_6^{k+1} = \frac{P_4}{z-4}.$$

Evaluating this equality at $z = 6$ gives

$$S_6 = \frac{P_4}{2}.$$

Since $P_6 P_4 = 0$ (independence of the spectral projections), we verify the remaining piece of part (iii) of Lemma 12.5.5 in that

$$P_6 S_6 = S_6 P_6 = 0.$$

Since $P_4^2 = P_4$, we obtain

$$S_6^{k+1} = (1/2)^{k+1} P_4,$$

thus giving the Laurent series of the resolvent about $\lambda = 6$, namely

$$\boxed{R_A(z) = \frac{D_6}{(z-6)^2} + \frac{P_6}{z-6} + P_4 \sum_{k=0}^{\infty} \frac{(-1)^k (z-6)^k}{2^{k+1}}}.$$

[This is how the Laurent series about λ can be written in terms of the spectral projections and D_λ. Answers to your homework problems should look like this.]

Using the geometric series one can verify that

$$\sum_{k=0}^{\infty} \frac{(-1)^k (z-6)^k}{2^{k+1}} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{6-z}{2} \right)^k = \frac{1}{2} \left(\frac{1}{1 - \frac{6-z}{2}} \right)$$
$$= \frac{1}{2} \left(\frac{2}{2-(6-z)} \right) = \frac{1}{2-(6-z)} = \frac{1}{z-4}.$$

What questions do you have?

To the next example.

Not as long of an Example (in lieu of 12.5.6). We compute the Laurent series

$$R_A(z) = \sum_{k=-\infty}^{\infty} A_k (z-2)^k$$

for the linear operator

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

about its eigenvalue $\lambda = 2$.

To this end we need to determine P_2, D_2, and S_2.

We computed previously that

$$R_A(z) = \frac{1}{z-2}\begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{(z-2)^2}\begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+ \frac{1}{(z-2)^3}\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{z-5}\begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The spectral projections are

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } P_5 = \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We also have

$$D_2 = A_{-2} = \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and by way of verification that

$$D_2^2 = \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_{-3}.$$

To find S_2 we have by part (iv) of Lemma 12.5.5 that

$$\sum_{k=0}^{\infty}(-1)^k(z-2)^k S_2^{k+1} = R_A(z) - \left(\frac{P_2}{z-2} + \frac{D_2}{(z-2)^2}\right)$$

and by part (v) of Lemma 12.5.5 that

$$P_2 R_A(z) = \frac{P_2}{z-2} + \frac{D_2}{(z-2)^2}.$$

Combining these gives

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k(z-2)^k S_2^{k+1} &= R_A(z) - P_2 R_A(z) = (I - P_2)R_A(z) \\ &= R_A(z)P_5 = \frac{P_5}{z-5},\end{aligned}$$

where we have used the completeness $P_2 + P_5 = I$ and part (iv) of Lemma 12.5.5 applied to $\lambda = 5$.

Evaluation of the equality at $z = 2$ gives $S_2 = -(1/3)P_5$.

The Laurent series for the resolvent of A around $\lambda = 2$ is

$$\boxed{R_A(z) = \frac{D_2^2}{(z-2)^3} + \frac{D_2}{(z-2)^2} + \frac{P_2}{z-2} - P_5 \sum_{k=0}^{\infty} \frac{(z-2)^k}{3^{k+1}}}.$$

Using the geometric series we can verify that

$$-\sum_{k=0}^{\infty} \frac{(z-2)^k}{3^{k+1}} = \frac{1}{z-5}.$$

We mentioned previously that $A \neq 2P_2 + 5P_5$, i.e.,

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \neq 2 \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + 5 \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

since A is not semisimple, but that something else was happening.

By including D_2, the spectral decomposition of A is

$$2P_2 + D_2 + 5P_2$$

because

$$
\begin{aligned}
&2P_2 + D_2 + 5P_5 \\
&= 2\begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \\
&= A.
\end{aligned}
$$

As we will see in the next section, the spectral decomposition of $A \in M_n(\mathbb{C})$ is

$$A = \sum_{\lambda \in \sigma(A)} \big(\lambda P_\lambda + D_\lambda\big).$$

Finding the spectral projections P_λ (or eigenprojections because as we will see $\mathscr{R}(P_\lambda) = \mathscr{E}_\lambda$) and, and as they will be called, the eigennilpotents D_λ, is achieved by the partial fraction decomposition of the resolvent R_A of A.

What questions do you have?