12.7 Spectral Mapping Theorem

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Recall the Semisimple Spectral Mapping Theorem 4.3.12 which states that for a semisimple $A \in M_n(\mathbb{C})$ and a polynomial $p \in \mathbb{C}[x]$, the set of eigenvalues of the the linear operator $p(A)$ is precisely

$$p(\sigma(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}.$$

We extend this result in two ways:

- to all linear operators $A \in M_n(\mathbb{C})$, and
- to all complex-valued functions f holomorphic on a simply connected open set containing the spectrum of a given linear operator.

This gives the Spectral Mapping Theorem.

Additionally we prove the uniqueness of the spectral decomposition of a linear operator

$$A = \sum_{\lambda \in \sigma(A)} \left(\lambda P_\lambda + D_\lambda\right)$$

for eigenprojections P_λ, with $\mathscr{R}(P_\lambda) = \mathscr{E}_\lambda$, and eigennilpotents D_λ.

This shows that the conclusion of Corollary 12.6.14,

$$f(A) = \sum_{\lambda \in \sigma(A)} \left[f(\lambda) P_\lambda + \sum_{k=1}^{m_\lambda - 1} a_{k,\lambda} D_\lambda^k \right],$$

is the spectral decomposition of $f(A)$.

Somewhere in the expression for $f(A)$ are the eigenprojections and the eigennilpotents for $f(A)$ for the eigenvalues of $f(A)$, and we will see exactly what they are by means of the Spectral Mapping Theorem.

Finally we use the spectral decomposition theory to develop the power method, a means of computing the eigenvector of a linear operator that has a dominant eigenvalue.

Theorem 12.7.1 (Spectral Mapping Theorem). For $A \in M_n(\mathbb{C})$, if f is holomorphic on an open disk containing $\sigma(A)$, then

$$\boxed{\sigma(f(A)) = f(\sigma(A))}.$$

Moreover, if $\mathrm{x} \in \mathbb{C}^n \setminus \{0\}$ is an eigenvector of A corresponding to $\lambda \in \sigma(A)$, then x is an eigenvector of $f(A)$ corresponding to $f(\lambda)$, i.e., for $\mathrm{x} \neq 0$,

$$A\mathrm{x} = \lambda\mathrm{x} \Rightarrow f(A)\mathrm{x} = f(\lambda)\mathrm{x}.$$

Second Reading Quiz Question. If $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, then $\sin(\lambda) \in \sigma(\sin(A))$.

True

The function $f(z) = \sin(z)$ is entire, hence holomorphic on the open set $\mathbb{C}$ containing $\sigma(A)$ for any $A \in M_n(\mathbb{C})$, so by the Spectral Mapping Theorem $\sin(\lambda)$ belongs to $\sigma(\sin(A))$.

Example (in lieu of 12.7.2). Recall that the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

are

$$\lambda_1 = 3,\ \xi_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \lambda_2 = -1,\ \xi_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The solution of the initial value problem $\mathrm{x}' = A\mathrm{x}$, $\mathrm{x}(0) = \mathrm{x}_0$ is

$$\mathrm{x}(t) = \exp(tA)\mathrm{x}_0.$$

For each constant $\alpha \in \mathbb{C}$, the function $f_\alpha(z) = \exp(\alpha z)$ is entire.

By the Spectral Mapping Theorem, we have

$$\sigma(f_\alpha(A)) = f_\alpha(\sigma(A)) = \{e^{\alpha\lambda_1}, e^{\alpha\lambda_2}\} = \{e^{3\alpha}, e^{-\alpha}\},$$

and the eigenvectors ξ_1, ξ_2 of A are eigenvectors of

$$f_\alpha(A) = \exp(\alpha A)$$

corresponding to $e^{3\alpha}$ and $e^{-\alpha}$ respectively.

Restricting $\alpha \in \mathbb{R}$, say $\alpha = t$, we obtain the eigenvalues

$$e^{3t} \text{ and } e^{-t}$$

with their corresponding eigenvectors ξ_1 and ξ_2 for $\exp(tA)$.

In particular, since $\exp(tA)\xi_2 = e^{-t}\xi_2$ for each $t \geq 0$, we have

$$\lim_{t\to\infty} \exp(tA)\xi_2 = \lim_{t\to\infty} e^{-t}\xi_2 = 0.$$

Similarly we would get

$$\lim_{t\to-\infty} \exp(tA)\xi_1 = \lim_{t\to-\infty} e^{3t}\xi_1 = 0.$$

The point of all of this is that we can compute these limits by means of the Spectral Mapping Theorem *without* explicitly computing $\exp(tA)$.

What questions do you have?

Recall the spectral decomposition

$$A = \sum_{\lambda \in \sigma(A)} \big(\lambda P_\lambda + D_\lambda\big).$$

The following gives its uniqueness.

Theorem 12.7.5. For $A \in M_n(\mathbb{C})$, if for each $\lambda \in \sigma(A)$ there is a projection $Q_\lambda \in M_n(\mathbb{C})$ and a nilpotent $C_\lambda \in M_n(\mathbb{C})$ satisfying

(i) $Q_\lambda Q_\mu = 0$ for all $\mu \in \sigma(A)$ with $\lambda \neq \mu$,

(ii) $Q_\lambda C_\lambda = C_\lambda Q_\lambda = C_\lambda$,

(iii) $Q_\mu C_\lambda = C_\lambda Q_\mu = 0$ for all $\mu \in \sigma(A)$ with $\mu \neq \lambda$,

(iv) $\sum_{\lambda \in \sigma(A)} Q_\lambda = I$, and

(v) $A = \sum_{\lambda \in \sigma(A)} \big(\lambda Q_\lambda + C_\lambda\big)$

then for each $\lambda \in \sigma(A)$ the projection Q_λ is the eigenprojection P_λ associated to A, and the nilpotent C_λ is the eigennipotent D_λ associated to A.

Sketch of Proof. For every $\mu \in \sigma(A)$ we have by item (v), the "spectral decomposition" of A, and items (i), (ii), and (iii) that

$$\begin{aligned} AQ_\mu &= \left(\sum_{\lambda \in \sigma(A)} (\lambda Q_\lambda + C_\lambda) \right) Q_\mu \\ &= \sum_{\lambda \in \sigma(A)} (\lambda Q_\lambda Q_\mu + C_\lambda Q_\mu) \\ &= \mu Q_\mu^2 + C_\mu Q_\mu \\ &= \mu Q_\mu + C_\mu \end{aligned}$$

This implies that

$$C_\mu = (A - \mu I)Q_\mu.$$

Since by Lemma 12.6.1

$$D_\mu = (A - \mu I)P_\mu,$$

we get $D_\mu = C_\mu$ by showing $P_\mu = Q_\mu$ for all $\mu \in \sigma(A)$.

The proofs of Lemma 12.6.7 and Theorem 10.6.9 that give $\mathscr{E}_\lambda = \mathscr{R}(P_\lambda)$ carry over to give

$$\mathscr{E}_\mu = \mathscr{R}(Q_\mu).$$

[Recall that in Section 10.6, Lemma 12.6.7 and Theorem 12.6.9 were consequences of the stated properties of projections and nilpotents, not specifically the projections and nilpotents that came from the resolvent.]

For $\mathrm{v} \in \mathbb{C}^n$ and $\lambda \in \sigma(A)$ we have

$$Q_\lambda \mathrm{v} \in \mathscr{E}_\lambda = \mathscr{R}(P_\lambda).$$

For $\mu \in \sigma(A) \setminus \{\lambda\}$ we have

$$P_\mu P_\lambda = 0$$

so that since $Q_\lambda \mathrm{v} \in \mathscr{R}(P_\lambda)$, we have

$$P_\mu Q_\lambda \mathrm{v} = 0.$$

Since $\mathscr{R}(Q_\mu) = \mathscr{E}_\mu$ and P_μ is a projection with the same range as Q_μ we have that

$$P_\mu Q_\mu \mathrm{v} = Q_\mu \mathrm{v}$$

for every $\mathrm{v} \in \mathbb{C}^n$ by Lemma 12.1.3.

Using item (v), the completeness of the projections Q_λ, $\lambda \in \sigma(A)$, we have for a fixed $\mu \in \sigma(A)$ that

$$\begin{aligned}
P_\mu \mathrm{v} &= P_\mu I \mathrm{v} \\
&= P_\mu \left(\sum_{\lambda \in \sigma(A)} Q_\lambda \mathrm{v} \right) \\
&= \sum_{\lambda \in \sigma(A)} P_\mu Q_\lambda \mathrm{v} \\
&= P_\mu Q_\mu \mathrm{v} \\
&= Q_\mu \mathrm{v}.
\end{aligned}$$

This implies that $P_\mu = Q_\mu$.

What questions do you have?

Theorem 12.7.6 (Mapping the Spectral Decomposition). Let $A \in M_n(\mathbb{C})$ and f be holomorphic on a simply connected open set U containing $\sigma(A)$. If for each $\lambda \in \sigma(A)$ we have the Taylor series

$$f(z) = f(\lambda) + \sum_{k=1}^{\infty} a_{n,\lambda}(z-\lambda)^k,$$

then

$$f(A) = \sum_{\lambda \in \sigma(A)} \left(f(\lambda)P_\lambda + \sum_{k=1}^{m_\lambda - 1} a_{k,\lambda} D_\lambda^k \right)$$

is the spectral decomposition of $f(A)$, i.e., for each $\nu \in \sigma(f(A))$ the eigenprojection for $f(A)$ is given by

$$\sum_{\mu \in \sigma(A), f(\mu)=\nu} P_\mu,$$

and the corresponding eigennilpotent D_ν is given by

$$\sum_{\mu \in \sigma(A), f(\mu)=\nu} \sum_{k=1}^{m_\mu - 1} a_{k,\mu} D_\mu^k.$$

Example (in lieu of 12.7.7). Find the eigenprojections and eigennilpotents of the square of

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

using the formulas in Theorem 12.7.6.

From the partial fraction decomposition of the entries of the resolvent $R_A(z)$ we obtain

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/9 \\ 0 & 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The spectrum of A is $\sigma(A) = \{1, -1, 2\}$ and the spectral decomposition is

$$A = P_1 + D_1 - P_{-1} + D_{-1} + 2P_2.$$

Since $f(z) = z^2$ is entire, we have by the Spectral Mapping Theorem that

$$\sigma(f(A)) = f(\sigma(A)) = \{1^2, (-1)^2, 2^2\} = \{1, 4\}.$$

What questions do you have before we proceed?

The eigenprojection for $f(A)$ corresponding to $\nu = 1 \in \sigma(f(A))$ is

$$\sum_{\mu \in \sigma(A), f(\mu)=1} P_\mu = P_1 + P_{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/9 \\ 0 & 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1/9 \\ 0 & 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenprojection for $f(A)$ corresponding $\nu = 4 \in \sigma(f(A))$ is

$$P_2.$$

To get the eigennilpotent for $f(A)$ corresponding to $\nu = 1$ we compute the Taylor series expansions of

$$f(z) = z^2$$

about $z = 1$ and about $z = -1$:

$$\begin{aligned} z^2 &= (1 + z - 1)^2 \\ &= (1 + (z - 1))^2 \\ &= 1 + 2(z - 1) + (z - 1)^2, \\ z^2 &= (-1 + z + 1)^2 \\ &= (-1 + (z + 1))^2 \\ &= 1 - 2(z + 1) + (z + 1)^2. \end{aligned}$$

Here we have

$$\begin{aligned} a_{1,1} &= 2, \\ a_{1,-1} &= -2. \end{aligned}$$

The eigennilpotent for $f(A)$ corresponding to $\nu = 1$ is

$$\sum_{\mu\in\sigma(A), f(\mu)=1} \sum_{k=1}^{m_\mu - 1} a_{k,\mu} D_\mu^k$$

$$= 2D_1 - 2D_{-1}$$

$$= \begin{bmatrix} 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can verify all of the eigenprojections and eigennilpotents for

$$f(A) = A^2$$

by directly squaring the spectral decomposition of A:

$$\begin{aligned} A^2 &= (P_1 + D_1 - P_{-1} + D_{-1} + 2P_2)(P_1 + D_1 - P_{-1} + D_{-1} + 2P_2) \\ &= P_1 + 2D_1 + P_{-1} - 2D_{-1} + 4P_2 \\ &= P_1 + P_{-1} + 2D_1 - 2D_{-1} + 4P_2, \end{aligned}$$

where we use

$$P_1^2 = P_1,\ P_{-1}^2 = P_{-1},\ P_2^2 = P_2,\ D_1^2 = 0,\ D_{-1}^2 = 0,$$

and

$$P_1 D_{-1} = 0,\ P_{-1} D_1 = 0, \mathrm{etc.}$$

What questions do you have?

Example (in lieu of 12.7.7). Use Theorem 12.7.6 to find the inverse of the invertible

$$A = \begin{bmatrix} -1 & 11 & -3 \\ -2 & 8 & -1 \\ -1 & 5 & 0 \end{bmatrix}.$$

Recall from last time that the spectral decomposition is

$$A = 2P_2 + D_2 + 3P_3$$

where

$$P_2 = \begin{bmatrix} 11 & -18 & -4 \\ 5 & -8 & -2 \\ 5 & -9 & -1 \end{bmatrix},\ D_2 = \begin{bmatrix} 7 & -7 & -7 \\ 3 & -3 & -3 \\ 4 & -4 & -4 \end{bmatrix},\ P_3 = \begin{bmatrix} -10 & 18 & 4 \\ -5 & 9 & 2 \\ -5 & 9 & 2 \end{bmatrix}.$$

To find the inverse of A we use Taylor series expansions of function $f(z) = z^{-1}$ about $\lambda \in \sigma(A) = \{2, 3\}$.

Since

$$f^{(l)}(z) = \frac{(-1)^l l!}{z^{l+1}},$$

the Taylor series of $f(z)$ about $\lambda \neq 0$ is

$$f(z) = \sum_{l=0}^{\infty} \frac{(-1)^l}{\lambda^{l+1}} (z-\lambda)^l.$$

We identify

$$a_{l,\lambda} = \frac{(-1)^l}{\lambda^{l+1}}.$$

By Theorem 12.7.6 we obtain

$$A^{-1} = f(A) = \sum_{\lambda \in \sigma(A)} \left(f(\lambda) P_\lambda + \sum_{l=1}^{m_\lambda - 1} a_{l,\lambda} D_\lambda^l \right)$$

$$= \frac{1}{2} P_2 - \frac{1}{4} D_2 + \frac{1}{3} P_3 = \begin{bmatrix} 5/12 & -5/4 & 13/12 \\ 1/12 & -1/4 & 5/12 \\ -1/6 & -1/2 & 7/6 \end{bmatrix}.$$

[Computed and verified by Maple.]

The power method is algorithm to find an eigenvector for certain type of linear operator on a finite dimensional vector space.

Definition. An eigenvalue $\lambda \in \sigma(A)$ is called semisimple if the geometric multiplicity of λ equals its algebraic multiplicity.

Definition. An eigenvalue $\lambda \in \sigma(A)$ is called dominant if for all $\mu \in \sigma(A) \setminus \{\lambda\}$ there holds

$$|\lambda| > |\mu|.$$

Theorem 12.7.8. For $A \in M_n(\mathbb{C})$, suppose that $1 \in \sigma(A)$ is semisimple and dominant. If $\mathrm{v} \in \mathbb{C}^n$ satisfies $P_1\mathrm{v} \neq 0$, then for any norm $\|\cdot\|$ on $\mathbb{C}^n$, there holds

$$\lim_{k\to\infty} \|A^k\mathrm{v} - P_1\mathrm{v}\| = 0.$$

It is HW (Exercise 12.34) to extend this result to when the dominant eigenvalue is something other than $\lambda = 1$.

First Reading Quiz Question. Geometrically what is the Power Method doing?

Remark. The power method consist in making a good initial guess v for an eigenvector corresponding to the dominant semisimple eigenvalue 1.

The "good" part of the initial guess is that $P_1\mathrm{v} \neq 0$, because by the semisimpleness of $\lambda = 1$ the image $P_1\mathrm{v} \in \mathscr{E}_1 \setminus \{0\}$ is an eigenvector.

Although v is not necessarily an eigenvector, its iterates $A^k\mathrm{v}$ converge in any matrix norm to the eigenvector $P_1\mathrm{v}$ and the rate of convergence is determined by the dominance of the dominant semisimple eigenvalue. [Draw a picture.]

What questions do you have?

Example (in lieu of 12.7.9). The linear operator

$$A = \begin{bmatrix} 1/4 & 3/4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

is semisimple with spectrum

$$\sigma(A) = \{1, 1/4\}$$

where $1/4$ has algebraic multiplicity 2.

This means that eigenvalue 1 is semisimple and dominant.

To find an eigenvector corresponding to eigenvalue 1 by the power method we start with the initial guess

$$\mathrm{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then

$$A\mathrm{v} = \begin{bmatrix} 1/4 & 3/4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1/4 \end{bmatrix},$$

$$A^2\mathrm{v} = \begin{bmatrix} 1 \\ 1 \\ 1/16 \end{bmatrix},\ A^3\mathrm{v} = \begin{bmatrix} 1 \\ 1 \\ 1/64 \end{bmatrix}, \ldots,$$

$$A^k\mathrm{v} = \begin{bmatrix} 1 \\ 1 \\ 1/4^k \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Theorem 12.7.8 says this limit is an eigenvector of A corresponding to the eigenvalue 1 and that it is the image of the eigenprojection of the initial guess.

From the partial fraction decomposition of the resolvent $R_A(z)$ we have

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } P_{1/4} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We verify the spectral decomposition

$$\begin{bmatrix} 1/4 & 3/4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} = A = P_1 + (1/4)P_{1/4}.$$

The limit of $A^k \mathrm{v}$ is indeed

$$P_1 \mathrm{v} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

What questions do you have?