Math 346 Lecture #14
8.3 Measure Zero and Measurability

Our notion of volume or measure λ gives the same value for the compact n-interval $[\mathrm{a}, \mathrm{b}]$ and the open n-interval

$$(\mathrm{a}, \mathrm{b}) = (a_1, b_1) \times \cdots \times (a_n, b_n).$$

This says that the missing faces of the open n-interval have measure zero.

8.3.1 Sets of Measure Zero

There are some basic properties we expect of the measure λ for subsets of $\mathbb{R}^n$ on which it is defined.

1. For A and B in $\mathbb{R}^n$, if $B \subset A$ then $\lambda(B) \leq \lambda(A)$. (This property is called monotonicity.) This says that subsets of sets of measure zero have measure zero. Monotonicity of λ suggests that if $(C_k)_{k=1}^\infty$ is a sequence of sets for which $C_{k+1} \subset C_k$ and $\lambda(C_k) \to 0$ as $k \to \infty$, then

$$\lambda\left(\bigcap_{k=1}^\infty C_k\right) = \lim_{k\to\infty} \lambda(C_k) = 0,$$

   which expresses the "continuity" of λ on decreasing sequences of sets whose measures approach 0.
2. For A and B in $\mathbb{R}^n$, not necessarily disjoint, there holds

$$\lambda(A \cup B) \leq \lambda(A) + \lambda(B).$$

   (This property, which by induction extends to finite unions, is called finite subadditivity.) This suggest that if $(C_k)_{k=1}^\infty$ is a sequence of sets, then

$$\lambda\left(\bigcup_{k=1}^\infty C_k\right) \leq \sum_{k=1}^\infty \lambda(C_k),$$

   a property called countable subadditivity. [What would happen if the sets C_k were pairwise disjoint? We could replace $\leq$ with $=$ in the countable subadditivity, giving a property called countably additivity which is the key defining property of a measure. Countable additivity implies finite additivity, i.e., if A and B are disjoint, then $\lambda(A \cup B) = \lambda(A) + \lambda(B)$.]
3. The empty set $\emptyset$ is a subset of $\mathbb{R}^n$. It should have measure zero, i.e., $\lambda(\emptyset) = 0$. (This property is called finiteness of the measure on at least one set, i.e., there exists a set A for which $\lambda(A) < \infty$.)

So far we only know how to compute the measure of n-intervals, but the properties listed above suggest how to define sets of measure zero using compact, partially open, or open n-intervals.

Definition 8.3.1. A set $A \subset \mathbb{R}^n$ has measure zero if for any $\epsilon > 0$ there exists a countable collection of n-intervals $(I_k)_{n=1}^\infty$ such that

$$A \subset \bigcup_{k=1}^\infty I_k \quad \text{and} \quad \sum_{k=1}^\infty \lambda(I_k) < \epsilon.$$

Proposition 8.3.2. The following hold.

(i) Any subset of a set of measure zero has measure zero.
(ii) A singleton subset, i.e., {x} for x $\in \mathbb{R}^n$, has measure zero.
(iii) A countable union of sets of measure zero has measure zero.

Proof. (i) Suppose A is a set of measure zero.

Then for all $\epsilon > 0$ there exists a countable collection of n-intervals $(I_k)_{k=1}^{\infty}$ such that

$$A \subset \bigcup_{k=1}^{\infty} I_k \text{ and } \sum_{k=1}^{\infty} \lambda(I_k) < \epsilon.$$

For a subset B of A we then have

$$B \subset \bigcup_{k=1}^{\infty} I_k \text{ and } \sum_{k=1}^{\infty} \lambda(I_k) < \epsilon.$$

This says that B has measure zero.

(ii) This is HW (Exercise 8.11).

(iii) Suppose that $(C_k)_{k=1}^{\infty}$ is a countable collection of sets of measure zero.

For each fixed k we have for $\epsilon > 0$ the existence of a countable collection of n-intervals $(I_{j,k})_{j=1}^{\infty}$ for which

$$C_k \subset \bigcup_{j=1}^{\infty} I_{j,k} \text{ and } \sum_{j=1}^{\infty} \lambda(I_{j,k}) < \frac{\epsilon}{2^k}.$$

The collection $(I_{j,k})_{j,k=1}^{\infty}$ is a collection collection of n-intervals for which

$$\bigcup_{k=1}^{\infty} C_k \subset \bigcup_{k=1}^{\infty} \left(\bigcup_{j=1}^{\infty} I_{j,k} \right)$$

and

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda(I_{j,k}) &< \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} \\ &= \frac{\epsilon}{2} \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \\ &= \frac{\epsilon}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} \\ &= \frac{\epsilon}{2} \left(\frac{1}{1 - 1/2} \right) \\ &= \epsilon, \end{aligned}$$

where we have used the geometric series with $r = 1/2$.

Thus the union of the countable many sets of measure zero has measure zero. □

Example 8.3.3. The Cantor middle thirds set $C \subset [0,1]$ has measure zero.

The construction of C starts with $C_0 = [0,1]$, removes the open middle third subinterval of C_0 to obtain $C_1 = [0,1/3] \cup [2/3,1] \subset C_0$.

The open middle thirds of the two subintervals in C_1 are removed to obtain

$$C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1] \subset C_1.$$

Continuing this pattern by induction we obtain $C_{k+1} \subset C_k$ where C_k consists of 2^k pairwise disjoint compact subintervals $I_{j,k}$, $j = 1,\ldots,2^k$, each of which has length $(1/3)^k$.

To apply the definition of measure zero, we declare $I_{j,k} = \emptyset$ for all $j > 2^k$.

The Cantor middle thirds set,

$$C = \bigcap_{k=0}^{\infty} C_k,$$

then has the properties of

$$C \subset C_k = \bigcup_{j=1}^{\infty} I_{j,k} \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda(I_{j,k}) = 2^k \left(\frac{1}{3}\right)^k = \left(\frac{2}{3}\right)^k.$$

Since $(2/3)^k$ goes to 0 as $k \to \infty$, we conclude that C is a set of measure zero.

Definition 8.3.4. For a nonempty $A \subset \mathbb{R}^n$, two function $f, g : A \to \mathbb{R}$ are said to be equal almost everywhere on A, written $f = g$ a.e. on A if the set

$$\{t \in A : f(t) \neq g(t)\}$$

has measure zero.

Example (in lieu of 8.3.5). Consider the functions $f, g : [0,1] \to \mathbb{R}$ defined by $f(t) = 0$ for all $t \in [0,1]$ and

$$g(t) = \begin{cases} 1 & \text{if } t \in C, \\ 0 & \text{if } t \in [0,1] \setminus C, \end{cases}$$

where C is the Cantor middle thirds set.

The functions f and g differ on C which has measure zero, so $f = g$ a.e. on $[0,1]$.

Proposition 8.3.6. For a nonempty $A \subset \mathbb{R}^n$, the relation $=$ a.e. on A is an equivalence relation on the set of all functions from A to $\mathbb{R}$.

The proof of this when $A = [\mathrm{a}, \mathrm{b}] \subset \mathbb{R}^n$ is HW (Exercise 8.12).

Definition 8.3.7. For a nonempty $A \subset \mathbb{R}^n$, we say that a sequence of functions $(f_k)_{k=1}^{\infty}$ from A to $\mathbb{R}$ converges almost everywhere on A if the set

$$\{t \in A : (f_k(t))_{k=1}^{\infty} \text{ does not converge}\}$$

has measure zero. If for almost all $t \in A$ the sequence $(f_k(t))_{n=1}^{\infty}$ converges to $f(t)$, then we write $f_k \to f$ a.e. on A.

Note. Convergence almost everywhere is about pointwise convergence. It does not depend on the norm on the space of functions in which the sequence is.

Example 8.3.8. For the functions $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(t) = n\chi_{[0,1/n]}(t)$$

the sequence $(f_n(0))_{n=1}^{\infty}$ does not converge because $f_n(0) = n \to \infty$.

However for all $t \in (0,1]$, the sequence $(f_n(t))_{n=1}^{\infty}$ converges to 0 because eventually $f_n(t) = 0$ for sufficient large n, i.e., for fixed $t \in (0,1]$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ there holds $1/n < t$, so that for all $n \geq N$ there holds $f_n(t) = 0$.

The sequence $(f_n)_{n=1}^{\infty}$ converges almost everywhere to the zero function on $[0,1]$.

8.3.2 Measurability

The regulated integral applies to functions in $\mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R})$, the closure (which is the completion) of $S([\mathrm{a},\mathrm{b}],\mathbb{R})$ with respect to the L^∞-norm, i.e., functions that are the uniform limits of step functions.

The Daniell-Lebesgue integral applies to functions in $L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$, the completion of $S([\mathrm{a},\mathrm{b}],\mathbb{R})$ with respect to the L^1-norm, i.e., functions that are equal almost everywhere to the pointwise limits of L^1-Cauchy sequences of step functions.

In both of these situations the functions we obtain are pointwise limits a.e. on $[\mathrm{a},\mathrm{b}]$ of step functions. This motivates the following definitions.

Definition 8.3.9. A function $f : [\mathrm{a},\mathrm{b}] \to \mathbb{R}$ is called measurable if there exists a sequence $(s_k)_{k=1}^{\infty}$ of step functions such that $s_k \to f$ a.e. on $[\mathrm{a},\mathrm{b}]$.

A set $A \subset [\mathrm{a},\mathrm{b}]$ is called measurable if its indicator or characteristic function χ_A is measurable.

Note 8.3.13. Measurable sets include compact n-intervals, bounded open sets, bounded closed sets, and countable unions and intersections of bounded open or bounded closed sets. For example the half-open half-closed interval $(0,1]$ is measurable because it is the countable union of closed intervals:

$$(0,1] = \bigcup_{k=1}^{\infty} [1/k, 0].$$

Said in another way, the characteristic function $\chi_{(0,1]}$ is the pointwise limit of the sequence of step functions $s_k = \chi_{[1/k,0]}$.

Definition 8.3.10. Suppose a nonempty subset $A \subset [\mathrm{a},\mathrm{b}]$ is measurable. If $f : A \to \mathbb{R}$ satisfies $f\chi_A \in L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$, then we write

$$\int_A f = \int_{[\mathrm{a},\mathrm{b}]} f\chi_A.$$

We define $L^1(A,\mathbb{R})$ to be the collection of functions $f : A \to \mathbb{R}$ for which $f\chi_A \in L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$.

We show through the next two results that the integral of $f \in L^1(A,\mathbb{R})$ is independent of the compact n-interval $[\mathrm{a},\mathrm{b}]$ that contains the measurable A.

Proposition 8.3.11. Suppose the measurable A is a subset of the compact n-intervals $[\mathrm{a},\mathrm{b}]$ and $[\mathrm{c},\mathrm{d}]$ where $[\mathrm{a},\mathrm{b}] \subset [\mathrm{c},\mathrm{d}]$. Then $f\chi_A \in L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ if and only if $f\chi_A \in L^1([\mathrm{c},\mathrm{d}],\mathbb{R})$. Moreover there holds

$$\int_{[\mathrm{a},\mathrm{b}]} f\chi_A = \int_{[\mathrm{c},\mathrm{d}]} f\chi_A.$$

Proof. Suppose $f\chi_A \in L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$.

Then there is a sequence $(s_n)_{n=1}^\infty$ of step functions on $[\mathrm{a},\mathrm{b}]$ such that $(s_n)_{n=1}^\infty$ is L^1-Cauchy on $[\mathrm{a},\mathrm{b}]$ and $s_n \to f\chi_A$ a.e. on $[\mathrm{a},\mathrm{b}]$.

Extending every s_n by zero to $[\mathrm{c},\mathrm{d}]$ gives step functions t_n that satisfy $t_n \to f\chi_A$ a.e. on $[\mathrm{c},\mathrm{d}]$.

From the definition of the integral of a step function (the finite linear combination in $\mathbb{R}$) we have for all $m,n \in \mathbb{N}$ that

$$\int_{[\mathrm{c},\mathrm{d}]} |t_n - t_m| = \int_{[\mathrm{a},\mathrm{b}]} |s_n - s_m| \ \text{ and } \ \int_{[\mathrm{c},\mathrm{d}]} t_n = \int_{[\mathrm{a},\mathrm{b}]} s_n.$$

The first of these implies that $(t_n)_{n=1}^\infty$ is L^1-Cauchy on $[\mathrm{c},\mathrm{d}]$, so that $f\chi_A = \lim t_n$ belongs to $L^1([\mathrm{c},\mathrm{d}],\mathbb{R})$.

The second implies that

$$\int_{[\mathrm{c},\mathrm{d}]} f\chi_A = \int_{[\mathrm{a},\mathrm{b}]} f\chi_A.$$

Now suppose that $f\chi_A \in L^1(\mathrm{c},\mathrm{d}],\mathbb{R})$.

Then there is a sequence $(t_n)_{n=1}^\infty$ of step functions on $[\mathrm{c},\mathrm{d}]$ such that $(t_n)_{n=1}^\infty$ is L^1-Cauchy on $[\mathrm{c},\mathrm{d}]$ and $t_n \to f\chi_A$ a.e. on $[\mathrm{c},\mathrm{d}]$.

The functions $s_n = t_n\chi_{[\mathrm{a},\mathrm{b}]}$ are step functions on $[\mathrm{a},\mathrm{b}]$.

We show that $(s_n)_{n=1}^\infty$ is L^1-Cauchy on $[\mathrm{a},\mathrm{b}]$.

Since $(t_n)_{n=1}^\infty$ is L^1-Cauchy, for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n,m \geq N$ there holds (on $[\mathrm{c},\mathrm{d}]$)

$$\|t_n - t_m\|_1 < \epsilon.$$

Computing the L^1-norm of $s_n - s_m$ on $[\mathrm{a},\mathrm{b}]$ we have for all $n,m \geq N$ that

$$\int_{[\mathrm{a},\mathrm{b}]} |s_n - s_m| = \int_{[\mathrm{a},\mathrm{b}]} |t_n - t_m|\chi_{[\mathrm{a},\mathrm{b}]} \leq \int_{[\mathrm{c},\mathrm{d}]} |t_n - t_m| < \epsilon.$$

Thus $(s_n)_{n=1}^\infty$ is L^1-Cauchy on $[\mathrm{a},\mathrm{b}]$.

Since $t_n \to f\chi_A$ a.e. on $[\mathrm{c},\mathrm{d}]$ and $A \subset [\mathrm{a},\mathrm{b}] \subset [\mathrm{c},\mathrm{d}]$, we have that

$$s_n = t_n\chi_{[\mathrm{a},\mathrm{b}]} \to f\chi_A\chi_{[\mathrm{a},\mathrm{b}]} = f\chi_A.$$

Thus $f\chi_A \in L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$.

Using $t_n \to f\chi_A$ on $[\mathrm{c}, \mathrm{d}]$ and $s_n = t_n \chi_{[\mathrm{a},\mathrm{b}]} \to f\chi_A$ on $[\mathrm{a}, \mathrm{b}]$, we have

$$\begin{aligned}
\int_{[\mathrm{a},\mathrm{b}]} f\chi_A &= \lim_{n\to\infty} \int_{[\mathrm{a},\mathrm{b}]} s_n \\
&= \lim_{n\to\infty} \int_{[\mathrm{a},\mathrm{b}]} t_n \chi_{[\mathrm{a},\mathrm{b}]} \\
&= \lim_{n\to\infty} \int_{[\mathrm{c},\mathrm{d}]} t_n \chi_{[\mathrm{a},\mathrm{b}]} \\
&= \int_{[\mathrm{c},\mathrm{d}]} f\chi_A \chi_{[\mathrm{a},\mathrm{b}]} \\
&= \int_{[\mathrm{c},\mathrm{d}]} f\chi_A.
\end{aligned}$$

This completes the proof. □

Corollary 8.3.12. Suppose A is a measurable subset of $[\mathrm{c}, \mathrm{d}] \cap [\mathrm{c}', \mathrm{d}']$. Then $f\chi_A \in L^1([\mathrm{c}, \mathrm{d}], \mathbb{R})$ if and only if $f\chi_A \in L^1([\mathrm{c}', \mathrm{d}'], \mathbb{R})$.

Proof. The intersection $[\mathrm{c}, \mathrm{d}] \cap [\mathrm{c}', \mathrm{d}']$ is a compact n-interval $[\mathrm{a}, \mathrm{b}]$.

We have that $[\mathrm{a}, \mathrm{b}] \subset [\mathrm{c}, \mathrm{d}]$ and $[\mathrm{a}, \mathrm{b}] \subset [\mathrm{c}', \mathrm{d}']$.

We apply Proposition 8.3.11 to these inclusions to obtain that $f\chi_A \in L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$ if and only if $f\chi_A \in L^1([\mathrm{c}, \mathrm{d}], \mathbb{R})$, with

$$\int_{[\mathrm{a},\mathrm{b}]} f\chi_A = \int_{[\mathrm{c},\mathrm{d}]} f\chi_A,$$

and $f\chi_A \in L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$ if and only if $f\chi_A \in L^1([\mathrm{c}', \mathrm{d}'], \mathbb{R})$ with

$$\int_{[\mathrm{a},\mathrm{b}]} f\chi_A = \int_{[\mathrm{c}',\mathrm{d}']} f\chi_A.$$

This completes the proof. □