Math 346 Lecture #17
8.6 Fubini's Theorem and Leibniz's Integral Rule

Fubini's Theorem – the switching of the order of the iterated integrals for the multivariate integral – is a consequence of passing the switching of the order of iterated integrals on step functions (which is easily shown) to L^1 functions by means of the Monotone Convergence Theorem.

A consequence of Fubini's Theorem is Leibniz's integral rule which gives conditions by which a derivative of a partial integral is the partial integral of a derivative, which is a useful tool in computation of multivariate integrals.

8.6.1 Fubini's Theorem

We fix some notation to aid in stating Fubini's Theorem.

Let $X = [\mathrm{a}, \mathrm{b}] \subset \mathbb{R}^n$ and $Y = [\mathrm{c}, \mathrm{d}] \subset \mathbb{R}^m$.

For $g \in L^1(X, \mathbb{R})$ we write the integral of g as

$$\int_X g(\mathrm{x})\ d\mathrm{x}.$$

For $h \in L^1(Y, \mathbb{R})$ we write the integral of h as

$$\int_Y h(\mathrm{y})\ d\mathrm{y}.$$

For $f \in L^1(X \times Y, \mathbb{R})$ we write the integral of f as

$$\int_{X \times Y} f(\mathrm{x}, \mathrm{y})\ d\mathrm{x}d\mathrm{y}.$$

Note. The measure $d\mathrm{x}d\mathrm{y}$ on $\mathbb{R}^{n+m}$ is not quite the "product" of the measures $\lambda_n = d\mathrm{x}$ on $\mathbb{R}^n$ and $\lambda_m = d\mathrm{y}$ on $\mathbb{R}^n$. The measure $d\mathrm{x}d\mathrm{y} = \lambda_{n+m}$ is the "completion" of the product of the measures $d\mathrm{x}$ and $d\mathrm{y}$, that is, the missing subsets of sets of measure zero are added and the product measure is extended.

For $f : X \times Y \to \mathbb{R}$ we define for each $\mathrm{x} \in X$ the function $f_{\mathrm{x}} : Y \to \mathbb{R}$ by

$$f_{\mathrm{x}}(\mathrm{y}) = f(\mathrm{x}, \mathrm{y}).$$

Theorem 8.6.1 (Fubini's Theorem). If $f \in L^1(X \times Y, \mathbb{R})$, then

(i) for almost all $\mathrm{x} \in X$, we have $f_{\mathrm{x}} \in L^1(Y, \mathbb{R})$,

(ii) the function $F : X \to \mathbb{R}$ defined by

$$F(\mathrm{x}) = \begin{cases} \displaystyle\int_Y f_{\mathrm{x}}(\mathrm{y})\ d\mathrm{y} & \text{if } f_{\mathrm{x}} \in L^1(Y, \mathbb{R}), \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $L^1(X, \mathbb{R})$, and

(iii) there holds

$$\int_{X\times Y} f(\mathrm{x},\mathrm{y})\ d\mathrm{x}d\mathrm{y} = \int_X F(\mathrm{x})\ d\mathrm{x} = \int_X \left(\int_Y f_{\mathrm{x}}(\mathrm{y})d\mathrm{y}\right) d\mathrm{x}.$$

The proof of this is in Chapter 9 (which we are skipping).

Remark 8.6.2. We call

$$\int_X \left(\int_Y f_{\mathrm{x}}(\mathrm{y})d\mathrm{y}\right) d\mathrm{x} = \int_X \left(\int_Y f(\mathrm{x},\mathrm{y})d\mathrm{y}\right) d\mathrm{x}$$

an iterated integral of f.

Nota Bene 8.6.3. The hypothesis $f \in L^1(X\times Y,\mathbb{R})$ cannot be weakened. Examples exist for which $f_{\mathrm{x}} \in L^1(Y,\mathbb{R})$ for all $\mathrm{x}\in X$ and $F\in L^1(X,\mathbb{R})$ but $f \notin L^1(X\times Y)$ and the two iterated integrals exists but differ in value.

Note. Each integral in an iterated integral can often be computed using the Fundamental Theorem of Calculus.

Example (in lieu 8.6.4). For $X=[0,\pi]$ and $Y=[1,2]$ the function $f: X\times Y\to\mathbb{R}$ defined by

$$f(x,y) = x\cos(xy)$$

is continuous on $X\times Y$ and thus belongs to $L^1(X\times Y,\mathbb{R})$.

By Fubini's Theorem and the Fundamental Theorem of Calculus we have

$$\begin{aligned}
\int_{X\times Y} f(x,y)\ dxdy &= \int_X \left(\int_Y f(x,y)\ dy\right)\ dx \\
&= \int_0^\pi \left(\int_1^2 x\cos(xy)\ dy\right)\ dx \\
&= \int_0^\pi \left(\sin(xy)\Big|_{y=1}^{y=2}\right)\ dx \\
&= \int_0^\pi \big(\sin(2x)-\sin(x)\big)\ dx \\
&= \left[\frac{-\cos(2x)}{2} + \cos(x)\right]_0^\pi \\
&= -\frac{1}{2} - 1 - \left(-\frac{1}{2}+1\right) \\
&= -2.
\end{aligned}$$

8.6.2 Interchanging the Order of Integration

Switching the roles of X and Y in Fubini's Theorem we get another iterated integral

$$\int_Y \left(\int_X f_{\mathrm{y}}(\mathrm{x})d\mathrm{x}\right) d\mathrm{y} = \int_Y \left(\int_X f(\mathrm{x},\mathrm{y})d\mathrm{x}\right) d\mathrm{y}$$

where $f_y : X \to \mathbb{R}$ is the function defined by $f_y(x) = f(x, y)$.

Proposition 8.6.5. If $f \in L^1(X \times Y, \mathbb{R})$, then function $\tilde{f} : Y \times X \to \mathbb{R}$ defined by $\tilde{f}(y, x) = f(x, y)$ belongs to $L^1(Y \times X, \mathbb{R})$, and there holds

$$\int_{Y\times X} \tilde{f}(y, x)\ dy dx = \int_{X\times Y} f(x, y)\ dx dy.$$

The proof of this is requested in Chapter 9 (as an exercise).

Corollary 8.6.6. If $f \in L^1(X \times Y, \mathbb{R})$, then

$$\int_X \left(\int_Y f_x(y)\ dy \right)\ dx = \int_{X\times Y} f(x, y)\ dx dy = \int_Y \left(\int_X f_y(x)\ dx \right)\ dy.$$

The proof of the Corollary follows immediately from Fubini's Theorem and Proposition 8.6.5.

Corollary 8.6.6 permits computing the integral of f over $X \times Y$ by either of the two iterated integrals. Often one of the iterated integrals is much easier to compute than the other.

Example (in lieu of 8.6.7). If $f(x, y) = g(x)h(y)$ for continuous functions $g : X \to \mathbb{R}$ and $h : Y \to \mathbb{R}$, then f is continuous on $X \times Y$, hence belongs to $L^1(X \times Y, \mathbb{R})$, so by the Fubini's Theorem we have

$$\begin{aligned} \int_{X\times Y} f(x, y)\ dx dy &= \int_X \left(\int_Y g(x)h(y)\ dy \right)\ dx \\ &= \int_X \left(g(x) \int_Y h(y)\ dy \right)\ dx \\ &= \left(\int_Y h(y)\ dy \right) \int_X g(x)\ dx \\ &= \left(\int_X g(x)\ dx \right) \left(\int_Y h(y)\ dy \right). \end{aligned}$$

By switching the order of integration we arrive at the same answer.

Example (in lieu of 8.6.8). For a bounded measurable set $S \subset \mathbb{R}^n \times \mathbb{R}^m$, choose a compact $(m + n)$-interval $X \times Y$ that contains S. For a measurable function $f : S \to \mathbb{R}$ that satisfies $f\chi_S \in L^1(X \times Y, \mathbb{R})$, we define the double integral of f over S by

$$\iint_S f\ dx dy = \int_{X\times Y} f\chi_S\ dx dy.$$

The function f is extended by zero outside of S to the complement $X \times Y - S$.

A sufficient condition for $f\chi_S \in L^1(X \times Y, \mathbb{R})$ is that f is continuous on S and that the boundary of S is piecewise differentiable, i.e., each boundary part of S is the graph of a differentiable function, and that $f : S \to \mathbb{R}$ is continuous. This means that the set on which the extended by zero function f is discontinuous is a measurable set of measure zero and therefore $f \in L^1(X \times Y, \mathbb{R})$.

Example. Consider the subset S of $\mathbb{R}^2$ given by

$$S = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : -1 \leq x \leq 1, -1 \leq y \leq \sqrt{1 - x^2}\right\}.$$

The set S has piecewise differentiable boundary, and as a compact subset of $\mathbb{R}^2$, is a measurable set contained in the compact 2-interval $X \times Y = [-1, 1] \times [-1, 1]$.

The top boundary of S is the graph of the differentiable function

$$b(x) = \sqrt{1 - x^2}$$

while the bottom of S is the graph of the differentiable function

$$a(x) = -1.$$

To compute the double integral of a continuous $f : S \to \mathbb{R}$ we can make use of variable upper and lower limits to account for χ_S in the inner integral of the iterated integral:

$$\begin{aligned} \iint_S (x, y)\ \mathrm{dxdy} &= \int_0^1 \left(\int_0^1 f(x, y)\chi_S\ \mathrm{dy} \right)\ \mathrm{dx} \\ &= \int_0^1 \left(\int_{a(x)}^{b(x)} f(x, y)\ \mathrm{dy} \right)\ \mathrm{dx}. \end{aligned}$$

The integrability of the inner integral will be justified by the upcoming Corollary of Leibniz's Integral Rule, while the replacement of the limits -1 and 1 of integration of the inner integral by $a(x)$ and $b(x)$ follows because $f\chi_S$ is zero outside of S which implies that the integral of f on $X \times Y - S$ is zero.

The double integral of $f(x, y) = x^2 y$ over S is

$$\begin{aligned} \iint_S f(x, y)\ dxdy &= \int_0^1 \left(\int_{a(x)}^{b(x)} x^2 y\ dy \right)\ dx \\ &= \int_0^1 \left(\frac{x^2}{2} \left[y^2 \right]_{y=-1}^{y=\sqrt{1-x^2}} \right)\ dx \\ &= \int_0^1 \left(\frac{x^2}{2} \left[(1 - x^2) - 1 \right] \right) dx \\ &= \int_0^1 \left(-\frac{x^4}{2} \right)\ dx \\ &= -\left[\frac{x^5}{10} \right]_{-1}^{1} \\ &= -\left(\frac{1}{10} - \left(\frac{(-1)^5}{10} \right) \right] \\ &= -\frac{2}{10} = -\frac{1}{5}. \end{aligned}$$

A similar approach would hold if the measurable S had the form

$$S = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : -1 \leq y \leq 1, -1 \leq x \leq \sqrt{1-y^2}\right\}$$

with the order of integration starting with x and then y.

8.6.3 Leibniz's Integral Rule

An important computational and theoretical tool for double integrals is Leibniz's integral rule, which, as a consequence of Fubini's Theorem, gives sufficient conditions by which differentiation can pass through the integral.

Theorem 8.6.9 (Leibniz's Integral Rule). For an open interval $X = (a, b) \subset \mathbb{R}$ and a compact interval $Y = [c, d] \subset \mathbb{R}$, if $f : X \times Y \to \mathbb{R}$ is continuous and the partial derivative $\frac{\partial f}{\partial x}$ is continuous on $X \times Y$, then the function

$$\psi(x) = \int_c^d f(x, y)\ dy$$

is differentiable on X, and the derivative of ψ is

$$\frac{d\psi(x)}{dx} = \int_c^d \frac{\partial f(x, y)}{\partial x}\ dy.$$

Proof. Fix $x_0 \in X$ and let $x \in X$ be arbitrary.

The compact interval with endpoints x_0 and x is a subset of $X = (a, b)$.

For each fixed $y \in [c, d] = Y$, the function $f_{\mathrm{y}}(x) = f(x, y)$ is continuous differentiable on the compact interval with endpoints x_0 and x, i.e., the derivative is continuous on the open interval with endpoints x_0 and x, and extends to a continuous function on the compact interval with endpoints x_0 and x.

Thus by part (ii) of the Fundamental Theorem of Calculus (Theorem 6.5.4) and Fubini's Theorem we have that

$$\begin{aligned}
\psi(x) - \psi(x_0) &= \int_c^d \big(f(x, y) - f(x_0, y)\big)\ dy \\
&= \int_c^d \left(\int_{x_0}^x \frac{\partial f(z, y)}{\partial z}\ dz\right)\ dy \\
&= \int_{x_0}^x \left(\int_c^d \frac{\partial f(z, y)}{\partial z}\ dy\right)\ dz.
\end{aligned}$$

For the function $g : X \to \mathbb{R}$ defined by

$$g(z) = \int_c^d \frac{\partial f(z, y)}{\partial z}\ dy$$

we have

$$\psi(x) - \psi(x_0) = \int_{x_0}^x g(z)\ dz.$$

To show the function ψ is differentiable on X we show that g is continuous on X and the apply part (i) of the Fundamental Theorem of Calculus.

Fix z_0 in the interior of the compact interval with endpoints x_0 and x.

The Cartesian product of the compact interval with endpoints x_0 and x and the compact interval $[c, d]$ is a compact subset of $\mathbb{R}^2$.

On this compact Cartesian product the continuous function $\partial f(z, y)/\partial z$ is uniformly continuous: for $\epsilon > 0$ there exists $\delta > 0$ such that for all points (z_1, y) and (z_0, y) in the Cartesian product satisfying

$$\|(z_1, y) - (z_0, y)\|_2 < \delta$$

there holds

$$\left| \frac{\partial f(z_1, y)}{\partial z} - \frac{\partial f(z_0, y)}{\partial z} \right| < \frac{\epsilon}{d - c}.$$

The inequality $\|(z_1, y) - (z_0, y)\|_2 < \delta$ implies that $|z_1 - z_0| < \delta$. (We have not used the full strength of the uniform continuity because we have put the same y in the two points.)

We use these inequalities to get the continuity of g on the compact interval with endpoints x_0 and x: when $|z_1 - z_0| < \delta$ we have

$$\begin{aligned}
|g(z_1) - g(z_0)| &= \left| \int_c^d \frac{\partial f(z_1, y)}{\partial z}\, dy - \int_c^d \frac{\partial f(z_0, y)}{\partial z}\, dy \right| \\
&= \left| \int_c^d \left(\frac{\partial f(z_1, y)}{\partial z} - \frac{\partial f(z_0, y)}{\partial z} \right) dy \right| \\
&\leq \int_c^d \left| \frac{\partial f(z_1, y)}{\partial z} - \frac{\partial f(z_0, y)}{\partial z} \right| dy \\
&\leq \int_c^d \frac{\epsilon}{d - c}\, dy \\
&= \epsilon.
\end{aligned}$$

The continuity of g on the compact interval with endpoints x_0 and x now implies by part (i) of the Fundamental Theorem of Calculus that

$$\frac{d}{dx} \int_{x_0}^{x} g(z)\, dz = g(x).$$

This implies, because

$$\psi(x) - \psi(x_0) = \int_{x_0}^{x} g(z)\, dz,$$

that ψ is differentiable at x in (a, b).

Since $x \in (a, b)$ is arbitrary, we have that ψ is differentiable on X where $d\psi(x)/dx = g(x)$.

Since

$$g(x) = \int_c^d \frac{\partial f(x, y)}{\partial x}\, dy,$$

we obtain the result. $\square$

Corollary 8.6.12. Let X and A be bounded open intervals in $\mathbb{R}$ and suppose $f : X \times A \to \mathbb{R}$ is continuous with continuous partial derivative $\partial f/\partial x$ on $X \times A$. If $a, b : X \to A$ are differentiable functions, then the function

$$\psi(x) = \int_{a(x)}^{b(x)} f(x,t)\ dt$$

is differentiable on X with derivative

$$\frac{d}{dx}\psi(x) = \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}\ dt - a'(x) f(x, a(x)) + b'(x) f(x, b(x)).$$

The proof of this is HW (Exercise 8.29 where a hint is given).

Corollary 8.6.12 justifies the integrability of the inner integral in the iterated integral in the Example (in lieu of 8.6.8) when S is a compact subset of $\mathbb{R}^2$ given by

$$S = \{(x,y) \in \mathbb{R}^2 : x \in I, a(x) \le y \le b(x)\}$$

where I is a compact interval and $a, b : I \to \mathbb{R}$ are differentiable functions with $a(x) \le b(x)$ for all $x \in I$, and f is extended by zero outside of S.