Math 346 Lecture #21
10.3 Parameterized Manifolds

The idea of a parameterized manifold is a higher-dimensional analogue of a smoothly parameterized curve.

We assume throughout this lecture that $(X, \|\cdot\|_X)$ is Banach space.

Definition 10.3.1. Let U be an open subset of $\mathbb{R}^m$. We say that $\alpha \in C^1(U, X)$ is a parameterized m-manifold in X if α is injective and at each point $\mathrm{u} \in U$ the derivative $D\alpha(u) \in \mathscr{B}(\mathbb{R}^m, X)$ is injective.

The image $M = \alpha(U) \subset X$ is a parametrized m-manifold.

A parameterized 2-manifold is called a parameterized surface.

A parameterized 1-manifold is called a parameterized curve and the condition that $D\alpha(\mathrm{u})$ be injective is equivalent to $D\alpha(\mathrm{u}) \neq 0$.

Remark 10.3.2. The injectivity of $D\alpha(\mathrm{u})$ at each $\mathrm{u} \in U$ implies that $m \leq \dim(X)$ because $D\alpha(\mathrm{u})(\mathbb{R}^m)$ is an m-dimensional subspace of X. In particular if $X = \mathbb{R}^n$ with $n \geq m$, then the rank of any matrix representation of $D\alpha(\mathrm{u})$ is m.

Example 10.3.3. (iii) For open $U \subset \mathbb{R}^m$ and $f \in C^1(U, \mathbb{R})$, the graph of f,

$$\{(\mathrm{u}, f(u)) \in \mathbb{R}^m \times \mathbb{R} : \mathrm{u} \in U\},$$

is a parameterized m-manifold because (1) the function $\alpha : U \to \mathbb{R}^m \times \mathbb{R}$ given by

$$\alpha(\mathrm{u}) = (\mathrm{u}, f(\mathrm{u}))$$

belongs to $C^1(U, \mathbb{R}^m \times \mathbb{R})$, (2) the function α is injective, i.e., if $\alpha(\mathrm{u}) = \alpha(\mathrm{v})$, then

$$(\mathrm{u}, f(\mathrm{u})) = (\mathrm{v}, f(\mathrm{v}))$$

which implies that $\mathrm{u} = \mathrm{v}$, and (3) $D\alpha(\mathrm{u})$ is injective for each $\mathrm{u} \in U$, i.e., $D\alpha(\mathrm{u})$ is the $(m+1) \times m$ matrix with the top $m \times m$ submatrix being I and the $(m+1)$-row being $Df(\mathrm{u})$, so that $D\alpha(\mathrm{u})$ has rank m.

10.3.1 Parameterizations and Equivalent Manifolds

Analogous to smooth parameterized curves, there are equivalence relations on parameterization manifolds, whose equivalent classes are called manifolds and have properties independent of the parameterization chosen.

Definition 10.3.4. Two parameterized m-manifolds $\alpha_1 : U_1 \to X$ and $\alpha_2 : U_2 \to X$ are called equivalent if there exists a diffeomorphism $\phi : U_1 \to U_2$ such that

(i) $\det(D\phi(\mathrm{u})) > 0$ for all $\mathrm{u} \in U_1$, and
(ii) $\alpha_2 = \alpha_1 \circ \phi$.

In this case we say that α_2 is a orientation-preserving reparameterization of α_1.

One can show that this equivalence is an equivalence relation.

Each equivalence class is called an oriented m-manifold, or if the dimension m is understood from the context, an oriented manifold.

If we replace condition (b) with $\det(D\phi(\mathrm{u})) \neq 0$, then by the continuity of $D\phi$ and the continuity of the determinant on the entries of the matrix, we either have $\det(D\phi(\mathrm{u})) > 0$ for all $\mathrm{u} \in U_1$ or $\det(D\phi(\mathrm{u})) < 0$ for all $\mathrm{u} \in U_1$.

When $\det(D\phi(\mathrm{u})) < 0$ for all $\mathrm{u} \in U_1$ we say the reparameterization ϕ is orientation-reversing.

One can show that the equivalence with the replaced condition (b) is an equivalence relation.

Each equivalence class for this equivalence relation is called an unoriented m-manifold or simply an m-manifold.

Example (in lieu of 10.3.5). For $U = (0, 2\pi) \times (0, \pi) \subset \mathbb{R}^2$, consider the C^1 map $\alpha : U \to \mathbb{R}^3$ defined by

$$\alpha(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi).$$

We show that α is injective.

Setting $\alpha(\theta_1, \varphi_1) = \alpha(\theta_2, \varphi_2)$ implies that $\cos\varphi_1 = \cos\varphi_2$.

Hence $\varphi_1 = \varphi_2$ because cos is injective on $(0, \pi)$.

With $\varphi_1 = \varphi_2$ we have $\sin\varphi_1 = \sin\varphi_2$ and since $\varphi_1 \in (0, \pi)$ that $\sin\varphi_1 \neq 0$.

From $\cos\theta_1 \sin\varphi_1 = \cos\theta_2 \sin\varphi_2$ and $\sin\theta_1 \sin\varphi_1 = \sin\theta_2 \sin\varphi_2$ we then get $\cos\theta_1 = \cos\theta_2$ and $\sin\theta_1 = \sin\theta_2$.

If θ_1 and θ_2 are in different quadrants, then either $\cos\theta_1 \neq \cos\theta_2$ or $\sin\theta_1 \neq \sin\theta_2$ would hold, a contradiction.

So θ_1 and θ_2 are in the same quadrant.

Monotonicity of cos and sin in the same quadrant implies that $\theta_1 = \theta_2$.

Thus α is injective.

The derivative

$$D\alpha(\theta, \varphi) = \begin{bmatrix} -\sin\theta \sin\varphi & \cos\theta \cos\varphi \\ \cos\theta \sin\varphi & \sin\theta \cos\varphi \\ 0 & -\sin\varphi \end{bmatrix}$$

has rank 2 for all $(\theta, \varphi) \in U$ because the term $-\sin\varphi \neq 0$ in the second column making the two columns linearly independent.

Thus α is a parameterized 2-manifold or surface in $\mathbb{R}^3$.

The image of α is a subset of the 2-sphere in $\mathbb{R}^3$ because

$$\begin{aligned}(\cos\theta \sin\varphi)^2 + (\sin\theta \sin\varphi)^2 + \cos^2\varphi &= (\cos^2\theta + \sin^2\theta)\sin^2\varphi + \cos^2\varphi \\ &= \sin^2\varphi + \cos^2\varphi \\ &= 1.\end{aligned}$$

In fact $\alpha(U)$ is almost all of the 2-sphere; the image is missing a set of measure zero, namely the smooth parameterized curve $C = \{(x, 0, \sqrt{1-x^2}) \in \mathbb{R}^3 : x \in [0,1]\}$ which is the longitudinal arc from the north pole to the sole pole in the xz-plane over $x \geq 0$.

For $W = (0,\pi) \times (0,\pi) \subset \mathbb{R}^2$, the C^1 function $\beta : W \to \mathbb{R}^3$ defined by

$$\beta(\xi,\eta) = (\cos 2\xi \sin\eta, \sin 2\xi \sin\eta, -\cos\eta)$$

is injective, the rank of

$$D\beta(\xi,\eta) = \begin{bmatrix} -2\sin 2\xi \sin\eta & \cos 2\xi \cos\eta \\ 2\cos 2\xi \sin\eta & \sin 2\xi \cos\eta \\ 0 & \sin\eta \end{bmatrix}$$

is 2 at every point $(\eta, \xi) \in W$, and $\beta(W) = \alpha(U)$.

The parameterized 2-manifold β is an orientating reversing reparameterization of the parameterized 2-manifold α because for the diffeomorphism $\phi : U \to W$ given by

$$\phi(\theta,\varphi) = (\theta/2, \pi - \varphi)$$

we have

$$\begin{aligned}(\beta \circ \phi)(\theta,\varphi) &= (\cos 2(\theta/2)\sin(\pi-\varphi), \sin 2(\theta/2)\sin(\pi-\varphi), -\cos(\pi-\varphi)) \\ &= (\cos\theta \sin\varphi, \sin\theta\sin\varphi, \cos\varphi) \\ &= \alpha(\theta,\varphi)\end{aligned}$$

because $\sin(\pi - \varphi) = \sin\varphi$ and $\cos(\pi-\varphi) = -\cos\varphi$, and

$$\det(D\phi(\theta,\varphi)) = \det\begin{bmatrix} 1/2 & 0 \\ 0 & -1 \end{bmatrix} = -1/2 < 0$$

for all $(\theta,\varphi) \in U$.

10.3.2 Tangent Spaces and Normals

For two parameterizations α and β of a manifold M, the derivatives $D\alpha$ and $D\beta$ are not usually the same, but as we show, their images are the same.

Definition 10.3.6. For a parameterized m-manifold $\alpha : U \subset \mathbb{R}^m \to M \subset X$, and a point $\mathrm{p} = \alpha(\mathrm{u}) \in M$, the tangent space $T_{\mathrm{p}}M$ of M at p is the image of the derivative $D\alpha(\mathrm{u}) \in \mathscr{B}(\mathbb{R}^m, X)$, i.e.,

$$T_{\mathrm{p}}M = \mathscr{R}(D\alpha(\mathrm{u})) = \{D\alpha(\mathrm{u})\mathrm{v} : \mathrm{v} \in \mathbb{R}^m\}.$$

Note. Because $D\alpha(\mathrm{u})$ is injective, if $\mathrm{v}_1, \ldots, \mathrm{v}_m$ is a basis for $\mathbb{R}^m$, then

$$D\alpha(\mathrm{u})\mathrm{v}_1, \ldots, D\alpha(\mathrm{u})\mathrm{v}_m$$

is a basis for $T_{\mathrm{p}}M$.

Example (in lieu of 10.3.7). Consider again, for $U = (0, 2\pi) \times (0, \pi) \subset \mathbb{R}^2$, the parameterized 2-manifold $\alpha : U \to M \subset \mathbb{R}^3$ given by

$$\alpha(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi).$$

The point $\mathrm{p} = (-1/\sqrt{2}, 0, 1/\sqrt{2}) \in M$ is $\alpha(\pi, \pi/4)$. Since

$$D\alpha(\pi, \pi/4) = \begin{bmatrix} 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{bmatrix}$$

a basis for $T_{\mathrm{p}}M$ is

$$D\alpha(\pi, \pi/4)\mathrm{e}_1 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \text{ and } D\alpha(\pi, \pi/4)\mathrm{e}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}.$$

For the reparameterization $\beta : W \to M$ given by

$$\beta(\xi, \eta) = (\cos 2\xi \sin\eta, \sin 2\xi \sin\eta, -\cos\eta)$$

where $W = (0, \pi) \times (0, \pi)$, the point $\mathrm{p} = (-1/\sqrt{2}, 0, 1/\sqrt{2})$ is $\beta(\pi/2, 3\pi/4)$, i.e., using the orientating reversing reparameterization $\phi : U \to W$, we have

$$\phi(\pi, \pi/4) = (\pi/2, \pi - \pi/4) = (\pi/2, 3\pi/4).$$

Since

$$D\beta(\pi/2, 3\pi/4) = \begin{bmatrix} 0 & 1/\sqrt{2} \\ -2/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}$$

a basis for $T_{\mathrm{p}}M$ is

$$D\beta(\pi/2, 3\pi/4)\mathrm{e}_1 = \begin{bmatrix} 0 \\ -2/\sqrt{2} \\ 0 \end{bmatrix} \text{ and } D\beta(\pi/2, 3\pi/4)\mathrm{e}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

One readily sees that $\mathscr{R}(D\beta(\pi/2, 3\pi/4))$ and $\mathscr{R}(D\alpha(\pi, \pi/4))$ are the same subspace.

Proposition 10.3.8. The tangent space $T_{\mathrm{p}}M$ is independent of the parameterization and of the orientation.

Proof. For open $U, W \in \mathbb{R}^m$, suppose $\alpha : U \to M$ and $\beta : W \to M$ are equivalent parameterizations of the unoriented manifold $M \subset X$.

Then there exists a diffeomorphism $\phi : U \to W$ such that $\beta \circ \phi = \alpha$. [The book writes $\phi = \beta^{-1} \circ \alpha$ which is correct because $\beta : W \to X$ is injective, so that the restriction $\beta : W \to M$ is invertible, but then the book differentiates $\beta^{-1} : M \to W$ which can be done, but we have only learned how to differentiate functions defined on open subsets of

Banach spaces, and unfortunately M is not in general an open subset of X. We proceed without using the derivative of β^{-1}.]

For $\mathrm{p} \in M$ there are unique $\mathrm{u} \in U$ and $\mathrm{w} \in W$ such that $\alpha(\mathrm{u}) = \mathrm{p} = \beta(\mathrm{w})$; in fact $\mathrm{w} = \phi(\mathrm{u})$.

Since $\beta \circ \phi = \alpha$ where β, ϕ, and α are C^1, we have

$$D\alpha(\mathrm{u}) = D(\beta \circ \phi)(\mathrm{u}) = D\beta(\phi(\mathrm{u}))D\phi(\mathrm{u}) = D\beta(\mathrm{w})D\phi(\mathrm{u}).$$

To show that $\mathscr{R}(D\beta(\mathrm{w})) = \mathscr{R}(D\alpha(\mathrm{u}))$, we take $\mathrm{x} \in \mathscr{R}(D\alpha(\mathrm{u}))$.

By the injectiveness of $D\alpha(\mathrm{u})$ there exists a unique $\mathrm{v} \in \mathbb{R}^m$ such $D\alpha(\mathrm{u})\mathrm{v} = \mathrm{x}$.

Then

$$\mathrm{x} = D\alpha(\mathrm{u})(\mathrm{v}) = D\beta(\mathrm{w})D\phi(\mathrm{u})\mathrm{v} = D\beta(\mathrm{w})\big(D\phi(\mathrm{u})\mathrm{v}\big),$$

which says that $\mathrm{x} \in \mathscr{R}(D\beta(\mathrm{w}))$.

This gives $\mathscr{R}(D\alpha(\mathrm{u})) \subset \mathscr{R}(D\beta(\mathrm{w}))$.

Now let $\mathrm{x} \in \mathscr{R}(D\beta(\mathrm{w}))$.

By the injectiveness of $D\beta(\mathrm{w})$ there exists a unique $\mathrm{v} \in \mathbb{R}^n$ such that $\mathrm{x} = D\beta(\mathrm{w})\mathrm{v}$.

Since $\phi : U \to W$ is a diffeomorphism, the map $D\phi(\mathrm{u}) \in \mathscr{B}(\mathbb{R}^m)$ is an isomorphism because differentiating $(\phi^{-1} \circ \phi)(\mathrm{z}) = \mathrm{z}$ gives

$$D\phi^{-1}(\phi(\mathrm{z}))D\phi(\mathrm{z}) = I,$$

which says that $D\phi(\mathrm{z})$ is invertible linear map.

Thus there exists a unique $\mathrm{y} \in \mathbb{R}^m$ such that $D\phi(\mathrm{u})\mathrm{y} = \mathrm{v}$.

This implies that

$$\mathrm{x} = D\beta(\mathrm{w})\mathrm{v} = D\beta(\mathrm{w})D\phi(\mathrm{u})\mathrm{y} = D\beta(\phi(\mathrm{u}))D\phi(\mathrm{u})\mathrm{y} = D\alpha(\mathrm{u})\mathrm{y}.$$

which says that $\mathrm{x} \in \mathscr{R}(D\alpha(\mathrm{u}))$.

This gives the other inclusion $\mathscr{R}(D\beta(\mathrm{w})) \subset \mathscr{R}(D\alpha(\mathrm{u}))$. □

Remark 10.3.9. The tangent space $T_{\mathrm{p}}M$ of a manifold M at $\mathrm{p} \in M$ is a vector subspace of X. One often draws the tangent space as a hyperplane touching and tangent to the manifold at the point p. This is technically incorrect since the hyperplane is not a vector subspace. We get around this technicality by translation: the hyperplane that touches and is tangent to M at P is the translation

$$\mathrm{p} + T_{\mathrm{p}}M = \{\mathrm{p} + \mathrm{x} : \mathrm{x} \in T_{\mathrm{p}}M\}.$$

Remark. In the case that M is a 2-manifold or surface in $X = \mathbb{R}^3$, we can use the cross product on the inner product space $\mathbb{R}^3$ with the standard inner product to construct a normal vector to M. Recall that in standard coordinates on $\mathbb{R}^3$, the cross product of two vectors $\mathrm{a} = (a_1, a_2, a_3)$ and $\mathrm{b} = (b_1, b_2, b_3)$ is the vector

$$\mathrm{a} \times \mathrm{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

The cross product a × b has the property that it is orthogonal to both a and b. The norm of a × b depends on the norms of a and b because

$$\|\mathrm{a} \times \mathrm{b}\| = \|\mathrm{a}\|\,\|\mathrm{b}\| \sin\theta$$

where $\theta \in [0, \pi)$ is the angle between a and b. Normalizing the cross product gives a vector of length one that no longer depends on the lengths of a and b.

Definition 10.3.10. For a surface $\alpha : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, the unit normal of M at $\mathrm{p} = \alpha(\mathrm{u})$ is the vector

$$\mathrm{N}(\mathrm{p}) = \frac{D\alpha(\mathrm{u})\mathrm{e}_1 \times D\alpha(\mathrm{u})\mathrm{e}_2}{\|D\alpha(\mathrm{u})\mathrm{e}_1 \times D\alpha(\mathrm{u})\mathrm{e}_2\|}.$$

Oftentimes we will write N instead of N(p).

Proposition 10.3.11. The unit normal N of a surface M in $\mathbb{R}^3$ depends only on the orientation of M. If the orientation of M is reversed, then N is negated.

Proof. For a surface $M \subset \mathbb{R}^3$ and open sets U, V in $\mathbb{R}^3$, let $\alpha : U \to M$ and $\beta : W \subset M$ be parameterizations of M.

Then there exists a diffeomorphism $\phi : U \to W$ such that $\beta \circ \phi = \alpha$ with $\det(D\phi(\mathrm{u})) \neq 0$ for all $\mathrm{u} \in U$.

For $\mathrm{p} \in M$ there exist unique $\mathrm{u} \in U$ and $\mathrm{w} \in W$ such that $\alpha(\mathrm{u}) = \mathrm{p} = \beta(\mathrm{w})$, i.e., $\mathrm{w} = \phi(\mathrm{u})$.

Since each of β, ϕ, and α is C^1, we have by the Chain Rule that

$$D\alpha(\mathrm{u}) = D(\beta \circ \phi)(\mathrm{u}) = D\beta(\phi(\mathrm{u}))D\phi(\mathrm{u}) = D\beta(\mathrm{w})D\phi(\mathrm{u}).$$

This implies for $i = 1, 2$ that

$$D\alpha(\mathrm{u})\mathrm{e}_i = D\beta(\mathrm{w})D\phi(\mathrm{u})\mathrm{e}_i.$$

Thus we have that

$$D\alpha(\mathrm{u})\mathrm{e}_1 \times D\alpha(\mathrm{u})\mathrm{e}_2 = D\beta(\mathrm{w})D\phi(\mathrm{u})\mathrm{e}_1 \times D\beta(\mathrm{w})D\phi(\mathrm{u})\mathrm{e}_2.$$

By a property of the cross product given in Proposition C 3.2 (in the Appendix of the book), there holds

$$D\beta(\mathrm{w})D\phi(\mathrm{u})\mathrm{e}_1 \times D\beta(\mathrm{w})D\phi(\mathrm{u})\mathrm{e}_2 = \det\big(D\phi(\mathrm{u})\big)\big(D\beta(\mathrm{w})\mathrm{e}_1 \times D\beta(\mathrm{w})\mathrm{e}_2\big).$$

This implies that

$$\begin{aligned}\frac{D\alpha(\mathrm{u})\mathrm{e}_1 \times D\alpha(\mathrm{u})\mathrm{e}_2}{\|D\alpha(\mathrm{u})\mathrm{e}_1 \times D\alpha(\mathrm{u})\mathrm{e}_2\|} &= \frac{\det\big(D\phi(\mathrm{u})\big)\big(D\beta(\mathrm{w})\mathrm{e}_1 \times D\beta(\mathrm{w})\mathrm{e}_2\big)}{\|\det\big(D\phi(\mathrm{u})\big)\big(D\beta(\mathrm{w})\mathrm{e}_1 \times D\beta(\mathrm{w})\mathrm{e}_2\big)\|} \\ &= \frac{\det\big(D\phi(\mathrm{u})\big)}{|\det\big(D\phi(\mathrm{u})\big)|}\,\frac{D\beta(\mathrm{w})\mathrm{e}_1 \times D\beta(\mathrm{w})\mathrm{e}_2}{\|D\beta(\mathrm{w})\mathrm{e}_1 \times D\beta(\mathrm{w})\mathrm{e}_2\|}\end{aligned}$$

The unit normal vectors are the same when $\det\bigl(D\phi(\mathrm{u})\bigr) > 0$, i.e., when β is an orientation preserving reparameterization of α.

The unit normal vectors are the negatives of each other when $\det\bigl(D\phi(\mathrm{u})\bigr) < 0$, i.e., when β is an orientation reversing reparameterization of α. $\square$

Example. Consider again the parameterizations $\alpha : U \to M$ and $\beta : W \to M$ of the surface M given by

$$\alpha(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$$

for $(\theta, \varphi) \in U = (0, 2\pi) \times (0, \pi)$ and

$$\beta(\xi, \eta) = (\cos 2\xi \sin\eta, \sin 2\xi \sin\eta, -\cos\eta)$$

for $(\xi, \eta) \in W = (0, \pi) \times (0, \pi)$.

The point $\mathrm{p} = (-1/\sqrt{2}, 0, 1/\sqrt{2}) = \alpha(\pi, \pi/4) = \beta(\pi/2, 3\pi/4)$, for which

$$D\alpha(\pi, \pi/4)\mathrm{e}_1 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \text{ and } D\alpha(\pi, \pi/4)\mathrm{e}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

and

$$D\beta(\pi/2, 3\pi/4)\mathrm{e}_1 = \begin{bmatrix} 0 \\ -2/\sqrt{2} \\ 0 \end{bmatrix} \text{ and } D\beta(\pi/2, 3\pi/4) = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

Computing the unit normals at p for each parameterization we have for α that

$$\mathrm{N} = \frac{(1/2, 0, -1/2)}{1/\sqrt{2}} = (1/\sqrt{2}, 0, -1/\sqrt{2}),$$

and for β that

$$\mathrm{N} = \frac{(-1, 0, 1)}{\sqrt{2}} = (-1/\sqrt{2}, 0, 1/\sqrt{2}).$$

As predicted by Proposition 10.3.11, the unit normals are the negatives of each other because β is an orientation reversing reparameterization of α.