Math 346 Lecture #25
11.2 Properties and Examples

Holomorphic functions satisfy all of the usual rules of differentiation, whether they are complex valued or more general complex Banach space X-valued. We will consider the case of $X = M_n(\mathbb{C})$ in Chapter 12.

Holomorphic functions also have a very close connection to convergent power series. The first part of this connection – that every convergent power series is holomorphic – we will see in this section.

Throughout this section $(X, \|\cdot\|_X)$ is a complex Banach space.

11.2.1 Basic Properties

Remark 11.2.1. For an open subset U of $\mathbb{C}$, a function $f : U \to X$ is continuous on U when f is holomorphic on U, because complex differentiability at a point implies continuity at that point (see Corollary 6.3.8).

Theorem 11.2.2. For U open in $\mathbb{C}$, suppose $f, g : U \to \mathbb{C}$ are holomorphic.

(i) For any constants $a, b \in \mathbb{C}$, the function $af + bg$ is holomorphic on U.

(ii) The product fg is holomorphic on U and

$$(fg)' = f'g + fg'.$$

(iii) If $g(z) \neq 0$ on U, then f/g is holomorphic on U and

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

(iv) For $k \in \mathbb{N}$ and $a_0, a_1, \ldots, a_k \in \mathbb{C}$, the polynomial $z \to a_0 + a_1 z + \cdots + a_k z^k$ is entire and its derivative is

$$z \to a_1 + 2a_2 z + \cdots + k a_k a^{k-1}.$$

(v) For $m, n \in \mathbb{N}$, $a_0, a_1, \ldots, a_n \in \mathbb{C}$, and $b_0, b_1, \ldots, b_m \in \mathbb{C}$, the rational function

$$z \to \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}$$

is holomorphic on the open set $\mathbb{C} \setminus \{\text{complex roots of the denominator}\}$.

Theorem 11.2.3 (Chain Rule). For open sets U, V in $\mathbb{C}$, if $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ are holomorphic, and $f(U) \subset V$, then $f \circ g : U \to \mathbb{C}$ is holomorphic and

$$(f \circ g)'(z) = f'(g(z))g'(z) \text{ for all } z \in U.$$

The proof of this follows from Theorem 6.4.7.

Proposition 11.2.4. For an open and path-connected U in $\mathbb{C}$, if $f : U \to X$ is holomorphic and $f'(z) = 0$ for all $z \in U$, then f is constant on U, i.e., there exists $\mathrm{x} \in X$ such that $f(z) = \mathrm{x}$ for all $z \in U$.

Proof. For any $z_1, z_2 \in U$ there is a smooth path $g : [0,1] \to U$ such that $g(0) = z_1$ and $g(1) = z_2$.

The composition $f \circ g$ is C^1 on $(0,1)$ and its derivative $(f \circ g)'$ is continous on $[0,1]$; these follows because f and g are both differentiable, and the hypothesis $f'(z) = 0$ for all $z \in U$ implies that the derivative $(f \circ g)'(t) = f'(g(t))g'(t)$ is the zero function which is continuous.

By the Fundamental Theorem of Calculus we have

$$f(z_2) - f(z_1) = f(g(1)) - f(g(0)) = \int_{[0,1]} (f \circ g)' = \int_{[0,1]} 0 = 0.$$

Since z_1, z_2 are arbitrary points of the path connected open U, we obtain that f is constant on U. □

11.2.2 Convergent Power Series are Holomorphic

We review some of the basic theory of convergent power series, some of which you saw in Math 341, and then prove that a convergent power series in a complex variable is holomorphic.

We look at more general power series as follows. For $a_k \in X$, $k = 0, 1, 2, \ldots$, and $z_0 \in \mathbb{C}$, a power series in X is

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

For each $r > 0$ and each $n = 0, 1, 2, \ldots$, the partial sum

$$f_n(z) = \sum_{k=0}^{n} a_k (z - z_0)^k$$

is a function belonging to the Banach space

$$(L^\infty(\overline{B(z_0, r)}, X), \| \cdot \|_\infty)$$

where

$$\|g\|_\infty = \sup_{z \in \overline{B(z_0,r)}} \|g(z)\|_X.$$

Convergence of the sequence of partial sums is always with respect to this Banach space, i.e., the topology of uniform convergence on compact sets.

A power series converges on an open set U if it converges on every compact subset of U.

Lemma 11.2.5 (Abel-Weierstrass Lemma). For a sequence $a_0, a_1, a_2, \cdots \in X$, if there exist an $R > 0$ and $M > 0$ such that for all $n = 0, 1, 2, \ldots$ there holds

$$\|a_n\|_X R^n \leq M,$$

then for any $0 < r < R$, the two series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \text{ and } \sum_{k=0}^{\infty} k a_k (z - z_0)^{k-1}$$

(the second being the formal term-by-term derivative of the first) both converge uniformly and absolutely on $\overline{B(z_0, r)} \subset \mathbb{C}$.

See the book for the proof.

Corollary 11.2.6. If a series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ diverges when $z = z_1$, then the series diverges at every $z \in \mathbb{C}$ that satisfies $|z - z_0| > |z_1 - z_0|$.

Proof. This is the contrapositive of the Abel-Weierstrass Lemma. □

Definition 11.2.7. Suppose a power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ converges on $\overline{B(z_0, r)}$ for some $r > 0$. The radius of convergence of the series is the supremum of the values of $R > 0$ for which the series converges uniformly on all compact subsets of the open $B(z_0, R)$.

The supremum is ∞ if the series converges uniformly on all compact subsets of $B(z_0, R)$ for all $R > 0$, and we say the radius of convergence is ∞.

Theorem 11.2.8. If a power series $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ converges uniformly on compact subsets of $B(z_0, R)$, then

(i) the function f is holomorphic on $B(z_0, R)$, and
(ii) the series $g(z) = \sum_{k=1}^{\infty} k a_k(z - z_0)^{k-1}$ converges uniformly on compact subsets of $B(z_0, R)$ and $f'(z) = g(z)$ on $B(z_0, R)$.

See the book for the proof.

Definition 11.2.9. For an open $U \subset \mathbb{C}$, a function $f : U \to X$ is called complex analytic (or simply analytic when there is no confusion with real analytic) if for all $z_0 \in U$ there exists $r > 0$ with $B(z_0, r) \subset U$ such that f can be written as a convergent power series on $B(z_0, r)$.

Remark 11.2.10. Any analytic function $f : U \to X$ is holomorphic on U by Theorem 11.2.8 part (i), and its derivative $f' : U \to X$ is analytic by Theorem 11.2.8 part (ii), and hence f' is holomorphic by Theorem 11.2.8 part (i). By in induction this means that every derivative $f^{(l)}$ is holomorphic, so that an analytic function is C^∞.

Example 11.2.11. (i) The power series

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

converges absolutely for any $z \in \mathbb{C}$ because the series

$$\sum_{k=0}^{\infty} \frac{|z|^k}{k!}$$

converges to $e^{|z|} < \infty$ for any $z \in \mathbb{C}$.

Thus the complex exponential function $\exp(z) = e^z$ is entire, and its derivative $f'(z)$ is itself.

(ii) The complex sine function is defined by the power series

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

which converges everywhere because

$$\sum_{k=0}^{\infty} \frac{|z|^{2n+1}}{(2n+1)!}$$

converges to $\sin|z| < \infty$.

The complex sine function $\sin(z)$ is entire and its derivative is the next function.

(iii) Similarly, the complex cosine function

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

is entire as well and its derivative is $-\sin(z)$.

Proposition 11.2.12 (Euler's Formula). For every $t \in \mathbb{C}$ there holds

$$\exp(it) = \cos(t) + i\sin(t).$$

See the book for the proof.

Example 11.2.13. We already learned by Theorem 11.2.8 part (i) that the complex exponential function is entire.

This means that the Cauchy-Riemnan equations should hold for the $\exp(z)$.

From Euler's Formula we have

$$\exp(z) = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y) = e^x \cos y + ie^x \sin y.$$

The functions $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$ satisfy

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x},$$

which are the Cauchy-Riemann equations.

Example 11.2.14. We can use the exponential map to define a function f from $\mathbb{C}$ to $M_n(\mathbb{C}) = \mathscr{B}(\mathbb{C}^n)$ by

$$f(z) = \sum_{k=0}^{\infty} \frac{(Az)^k}{k!}.$$

This series converges everywhere because

$$\sum_{k=0}^{\infty} \frac{\|A\|^k |z|^k}{k!} = \exp(\|A\|\,|z|) < \infty.$$

The function f is holomorphic with derivative

$$f'(z) = \sum_{k=1}^{\infty} k\frac{A^k z^{k-1}}{k!} = A\sum_{k=1}^{\infty} \frac{(Az)^{k-1}}{(k-1)!} = A\sum_{k=0}^{\infty} \frac{(Az)^k}{k!} = A\exp(Az).$$