Math 346 Lecture #26
11.3 Contour Integrals

A contour or path in $\mathbb{C}$ is a piecewise-smooth parameterized curve Γ in $\mathbb{C}$, i.e., the concatenation of a finite number of smooth parameterized curves.

Recall that a smooth parameterized curve is the image of an injective C^1 function with injective derivative.

We study the line integrals of holomorphic functions over contours.

Throughout we assume $(X, \|\cdot\|_X)$ is a complex Banach space.

11.3.1 Contour Integration

Definition 11.3.1. A parameterized contour or path $\Gamma \subset \mathbb{C}$ is a piecewise smooth curve in $\mathbb{C}$, that is, there are finitely many smoothly parameterized curves $\gamma_j : [a_j, b_j] \to \mathbb{C}$, $j = 1, \ldots, n$ such that $\gamma_{j+1}(a_{j+1}) = \gamma_j(b_j)$ for all $j = 1, \ldots, n-1$. We often write $\Gamma = \gamma_1 + \cdots + \gamma_n$ or

$$\Gamma = \sum_{j=1}^{n} \gamma_j.$$

The parameterized contour $-\Gamma$ is the parameterized contour Γ traversed in the opposite direction.

Definition 11.3.2. For an open set $U \subset \mathbb{C}$, let $f : U \to X$ be a continuous function and $\Gamma = \sum_{j=1}^{n} \gamma_i$ a contour in U. The contour integral of f over Γ is the element of X given by

$$\int_\Gamma f\ dz = \sum_{j=1}^{n} \int_{a_j}^{b_j} f(\gamma_j(t))\gamma_j'(t)\, dt \in X.$$

Remark 11.3.3. A contour integral can be computed also as a line integral in $\mathbb{R}^2$. By writing $\gamma_j(t) = x_j(t) + iy_j(t)$ for each $j = 1, \ldots, n$, the contour integral is

$$\int_\Gamma f\ dz = \sum_{j=1}^{n} \int_{a_j}^{b_j} f(\gamma_j(t))\big(x_j'(t) + iy_j'(t)\big)\, dt.$$

Remark 11.3.4. A contour integral does not usually represent area or volume, but often represents work done or energy. As we will see, contour integrals are an important tool for analyzing holomorphic functions.

Lemma 11.3.5. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the circle centered at $z_0 \in \mathbb{C}$ with radius $r > 0$, i.e.,

$$\gamma(\theta) = z_0 + re^{i\theta} = z_0 + r(\cos\theta + i\sin\theta).$$

For any $n \in \mathbb{Z}$ there holds

$$\int_\gamma (z - z_0)^n = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Setting $f(z) = (z - z_0)^n$ we have

$$\int_\gamma f \; dz = \int_0^{2\pi} (re^{i\theta})^n (ire^{i\theta}) \; d\theta = ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \; d\theta$$

When $n \neq -1$ we have

$$\int_0^{2\pi} e^{i(n+1)\theta} \; d\theta = \frac{1}{i(n+1)} \left[e^{i(n+1)\theta} \right]_0^{2\pi} = 0,$$

and thus the line integral of f over γ is 0.

When $n = -1$ we have

$$\int_0^{2\pi} e^0 \; d\theta = \int_0^{2\pi} 1 \; d\theta = 2\pi,$$

and thus the line integral of f over γ is $2\pi i$. □

Remark 11.3.6. Since a contour integral is a line integral in $\mathbb{R}^2$, the value of the contour integral is independent of the parameterization of the contour up to the orientation. For an oriented curve P we write the line integral of f over P as

$$\int_P f(z) \; dz,$$

where to actually compute the line integral requires a parameterization of P. If P is a simple closed curve, but no parameterization for P is given, the standing assumption is to assume the positive orientation for P, and instead of $\int_P f(z) \; dz$ we write

$$\oint_P f(z) \; dz.$$

We now state and prove the Fundamental Theorem of Calculus for Contour Integrals with an extra hypothesis that is missing in the book.

Theorem 11.3.7. If $\Gamma = \sum_{j=1}^n \gamma_j$ is a contour with start point z_0 and end point z_1, F is a holomorphic function on an open set $U \subset \mathbb{C}$ containing Γ, and F' is continuous on U, then

$$\int_\Gamma F'(z) \; dz = F(z_1) - F(z_0).$$

Proof. For each smooth $\gamma_j : [a_j, b_j] \to \mathbb{C}$ we have

$$\begin{aligned} \int_{\gamma_j} F'(z) \; dz &= \int_{a_j}^{b_j} F'(\gamma_j(t))\gamma_j'(t) \; dt \\ &= \int_{a_j}^{b_j} \frac{d}{dt} \big(F \circ \gamma_j\big)(t) \; dt \\ &= (F \circ \gamma_j)(t) \Big|_{a_j}^{b_j} \\ &= F(\gamma_j(b_j)) - F(\gamma_j(a_j)), \end{aligned}$$

where we have used Theorem 6.5.4, our Fundamental Theorem of Calculus for single-variable Banach space valued integration. [We need $F \circ \gamma$ to be C^1 to apply Theorem 6.5.4, so the assumption on the continuity of F' is required.]

Since $\gamma_{j+1}(a_{j+1}) = \gamma_j(b_j)$ for all $j = 1, \ldots, n-1$, we obtain

$$\int_\Gamma F'(z)\ dz = \sum_{j=1}^n \int_{a_j}^{b_j} F'(z)\ dz = \sum_{j=1}^n \big(F(\gamma_j(b_j)) - F(\gamma_j(a_j))\big).$$

This sum collapses since $\gamma_{j+1}(b_{j+1}) = \gamma_j(a_j)$ to $F(\gamma_n(b_n)) - F(\gamma_1(a_1))$.

Since $\gamma_n(b_n) = z_1$ and $\gamma_1(a_1) = z_0$, we obtain the result. □

Remark. Theorem 11.3.7 still holds if the hypothesis of continuity of F' on U is removed because of a result of Goursat who proved that the complex derivative of a holomorphic function is continuous. We will obtain continuity of F' as a consequence of the Cauchy-Goursat Theorem. We will state in the Corollary of Theorem 11.3.7 the assumption of continuity of F' on U, even though we know it can be removed.

Remark 11.3.8. Theorem 11.3.7, the Fundamental Theorem of Calculus for Line Integrals implies that line integrals of derivatives of holomorphic functions are independent of path, depending only on the start and end points of the path.

Corollary 11.3.9. If Γ is a closed contour in $\mathbb{C}$, $F : U \to X$ is a holomorphic on an open U in $\mathbb{C}$, and F' is continuous on U, then

$$\int_\Gamma F'(z)\ dz = 0.$$

Remark. The assumption in Corollary 11.3.9 of Γ being a closed contour in $\mathbb{C}$ allows for Γ to have self-intersections. The closed contour Γ does not have to be a simple closed curve. This is why we did not use the integral symbol $\oint$ in Corollary 11.3.9; we are not assuming Γ is a simple closed curve.

Example 11.3.10. We know from Lemma 11.3.5 that

$$\oint_\gamma (z - z_0)^n = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{otherwise} \end{cases}$$

for γ the unit circle with positive orientation.

We arrive at the same conclusion by Corollary 11.3.9 when $n \neq -1$ because

$$(z - z_0)^n = \frac{d}{dz}\left(\frac{(z - z_0)^{n+1}}{n+1}\right),$$

that is, for

$$F(z) = \frac{(z - z_0)^{n+1}}{n+1}$$

we have

$$F'(z) = (z - z_0)^n,$$

and so

$$\oint_\gamma (z-z_0)^n \ dz = \oint_\gamma F'(z) \ dz = 0.$$

Now you would like to think and hope it works that there is complex log function for which

$$\frac{d}{dz}\log(z-z_0) = (z-z_0)^{-1},$$

which would imply by Corollary 11.3.9 that the line integral $\int_\gamma (z-z_0)^{-1} \ dz = 0$.

But this cannot be, since we have already directly computed the line integral in the case of $n=-1$ and did not get zero for the answer.

This is telling us that there is no way to define the complex log function on a simple closed contour that encloses the origin.

The predicament is that while the real log function is the inverse of the bijective real exponential function, the complex exponential function is not injective, i.e.,

$$\begin{aligned}\exp(0+2\pi i k) &= e^0(\cos(2\pi k) + i\sin(2\pi k)) = 1 \\ &= e^0(\cos(0)+i\sin(0)) = \exp(0+i0) = \exp(0)\end{aligned}$$

for all $k \in \mathbb{Z}$; so we cannot define the complex log function as the inverse of the complex exponential function.

Thus there is no antiderivative of $(z-z_0)^{-1}$ on an open neighbourhood of a simple closed curve enclosing z_0.

11.3.2 The Cauchy-Goursat Theorem

For a contour Γ in $\mathbb{C}$ and a holomorphic function $f : U \to X$ defined on an open set U containing Γ, applying of the Fundamental Theorem of Line Integrals (Theorem 11.3.7) to compute

$$\int_\Gamma f(z) \ dz,$$

depends on finding an antiderivative F of f, i.e., $F' = f$ on U. But as we have seen in Example 11.3.10, not every holomorphic function f on U has an antiderivative F on U. Two other fundamental results in complex analysis, the Cauchy-Gourset Theorem and the Residue Theorem, give the tools needed to compute $\int_\Gamma f(z) \ dz$ without finding an antiderivative F of f on U. We will see the Residue Theorem in Section 11.7.

Theorem 11.3.11 (Cauchy-Goursat Theorem). *For a simply connected open set U in $\mathbb{C}$, if $f : U \to X$ is holomorphic then for any simple closed contour Γ in U there holds*

$$\oint_\Gamma f(z) \ dz = 0.$$

Remark 11.3.12. The Cauchy-Goursat Theorem and Corollary 11.3.9 each give different hypotheses on f and Γ for which

$$\int_\Gamma f(z) \ dz = 0.$$

The Cauchy-Goursat Theorem does not require finding an antiderivative of f as Corollary 11.3.9 does. Furthermore, the Cauchy-Gourset Theorem restricts U to being a simply connected open set and Γ a simple closed curve in U, whereas Corollary 11.3.9 only requires that U be open (not assumed simply connected) and that Γ be a closed curve (not assumed simple).

Remark 3.1.13. The proof of the Cauchy-Gourset Theorem, found in Subsection 11.3.3 (a starred subsection which we skip), is long and tedious because it does not assume the derivative of a holomorphic function is continuous.

If we assume that the derivative of a holomorphic function f is continuous, and we assume that $f : U \to \mathbb{C} \cong \mathbb{R}^2$, then we can use the Cauchy-Riemann equations and Green's Theorem to prove the Cauchy-Gourset Theorem.

With these assumptions we have $f(x + iy) = u(x, y) + iv(x, y)$ where $u, v : U \to \mathbb{R}$ are C^1. For a simply connected open U, and Γ a contour in U with positive orientation, and R the closure of the interior of Γ, we have by Green's Theorem and the Cauchy-Riemann equations that

$$\begin{aligned}
\oint_\Gamma f(z)\ dz &= \oint_\Gamma [u + iv](dx + idy) \\
&= \oint_\Gamma (u + iv)dx + (-v + iu)dy \\
&= \iint_R \left[\frac{\partial}{\partial x}(-v + iu) - \frac{\partial}{\partial y}(u + iv) \right]\ dxdy \\
&= \iint_R \left[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y} \right]\ dxdy \\
&= \iint_R \left[-\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \right]\ dxdy \\
&= \iint_R 0\ dxdy = 0.
\end{aligned}$$

The proof of the Cauchy-Goursat Theorem assumes only that f is holomorphic on a simply connected open set U and that $f : U \to X$ for a complex Banach space X.

Example 11.3.14. The function $z \to (z - z_0)^n$ is entire for each $n \geq 0$. Since $\mathbb{C}$ is simply connected, the Cauchy-Gourset Theorem implies whenever $n \geq 0$ that

$$\oint_\Gamma (z - z_0)^n\ dz = 0$$

for any simple closed contour Γ in $\mathbb{C}$.

However, we cannot apply the Cauchy-Gourset Theorem to the function $z \to (z - z_0)^n$ when $n < 0$ for any simple closed curve Γ that encloses z_0, because the function is not holomorphic on a simply connected open set U containing the interior of Γ.

The function $z \to (z - z_0)^n$ fails to be holomorphic at z_0 when $n < 0$, which is to say that this function is holomorphic on the punctured plane $\mathbb{C} \setminus \{0\}$.

Computing the line integral of function holomorphic except at finitely many points in the interior of a simple closed curve without having to parameterize the simple closed curve is done by the tool of the Residue Theorem.

Remark 11.3.15. The Cauchy-Goursat Theorem implies that contour integrals of any holomorphic function are path independent on a simply connected open set.

That is, if we have two contours $\gamma_i : [a_i, b_i] \to \mathbb{C}$, $i = 1, 2$, lying in a simply connected open set such that $\gamma_1(a_1) = \gamma_2(a_2)$ (same starting point) and $\gamma_1(b_1) = \gamma_2(b_2)$ (same ending point), and the concatenation $\gamma_1 - \gamma_2$ forms a simple closed contour, then the Cauchy-Goursat Theorem implies

$$0 = \int_{\gamma_1 - \gamma_2} f(z)\ dz = \int_{\gamma_1} f(z)\ dz - \int_{\gamma_2} f(z)\ dz$$

so that

$$\int_{\gamma_1} f(z)\ dz = \int_{\gamma_2} f(z)\ dz.$$

Example 11.3.16. We computed in Lemma 11.3.5 that

$$\oint_{\gamma_0} \frac{1}{z - z_0}\ dz = 2\pi i$$

for the positively oriented circle $\gamma_0(\theta) = z_0 + re^{i\theta}$ for any $r > 0$.

For any simple closed contour γ_1 enclosing z_0 there is a small enough $r > 0$ such that γ_0 is contain completely in the interior of γ_1.

By using a "cut" σ from the starting point on γ_0 to the starting point on γ_1, we can form a simply connected open set with boundary

$$\Gamma = \gamma_1 + \sigma - \gamma_0 - \sigma.$$

By the Cauchy-Gourset Theorem we obtain

$$\begin{aligned} 0 &= \oint_\Gamma \frac{1}{z - z_0}\ dz \\ &= \int_{\gamma_1} \frac{1}{z - z_0}\ dz + \int_\sigma \frac{1}{z - z_0}\ dx - \int_{\gamma_0} \frac{1}{z - z_0}\ dz - \int_\sigma \frac{1}{z - z_0}\ dz \\ &= \int_{\gamma_1} \frac{1}{z - z_0}\ dz - \int_{\gamma_0} \frac{1}{z - z_0}\ dz. \end{aligned}$$

This implies that

$$\int_{\gamma_1} \frac{1}{z - z_0}\ dz = \int_{\gamma_0} \frac{1}{z - z_0}\ dz,$$

which is to say that for any simple closed contour γ_1 enclosing z_0 the line of $f(z) = (z - z_0)^{-1}$ over γ_1 is $2\pi i$.