Math 541 Lecture #1
I.1: Topological Spaces
I.4: Bases, axioms of countability, and product topologies, Part I

1: Topological Spaces. We review the basic definitions and concepts of topological spaces that we will need later for infinite sets of functions.

As a visual keep in mind the familiar topological notion of open set in $\mathbb{R}^N$.

A collection of subsets $\mathcal{U}$ of a set X defines a **topology** on X if the following three axioms hold:

(i) the empty set $\emptyset$ and X belong to $\mathcal{U}$,
(ii) the union of any collection of sets in $\mathcal{U}$ belongs to $\mathcal{U}$, and
(iii) the intersection of finitely many elements of $\mathcal{U}$ belongs to $\mathcal{U}$.

The **weakest or coarsest or trivial** topology on X is $\mathcal{U} = \{\emptyset, X\}$, while the **strongest or finest or discrete** topology on X is $\mathcal{U} = \mathcal{P}(X) = 2^X$ (the power set of X).

A **topological space** is a pair $\{X;\mathcal{U}\}$ where $\mathcal{U}$ is a topology on X.

A set $\mathcal{O}$ of a topological space $\{X;\mathcal{U}\}$ is **open** if $\mathcal{O} \in \mathcal{U}$.

Any union of open sets is open by axiom (ii).

A set C of a topological space $\{X;\mathcal{U}\}$ is **closed** if its complement $C^c = X - C$ is open.

The intersection of any collection of closed sets $\{C_\alpha : \alpha \in I\}$ (for some index set I) is closed by axiom (ii) and DeMorgan's Law,

$$X - \bigcap_{\alpha\in I} C_\alpha = X \cap \left(\bigcap_{\alpha\in I} C_\alpha\right)^c = X \cap \left(\bigcup_{\alpha\in I} C_\alpha^c\right) = \bigcup_{\alpha\in I} X \cap C_\alpha^c = \bigcup_{\alpha\in I}(X - C_\alpha),$$

while the union of finitely many closed sets $C_1, \ldots, C_n$ is closed by axiom (iii) and DeMorgan's Law,

$$X - (C_1 \cup \cdots \cup C_n) = (X - C_1) \cap \cdots \cap (X - C_n).$$

No matter the topology chosen for X, the sets $\emptyset$ and X are both open and closed (or clopen) by axiom (i).

An **open neighbourhood** (open nhbd for short) of a subset A of X is any open set $\mathcal{O}$ for which $A \subset \mathcal{O}$.

A open nbhd of a singleton or point $x \in X$ is any open set $\mathcal{O}$ for which $x \in \mathcal{O}$.

Fact: a subset $\mathcal{O}$ of X is open if and only if $\mathcal{O}$ is an open nhbd of any of its points.

A point $x \in A$ is an **interior point of** A if there is an open set $\mathcal{O}$ such that

$$x \in \mathcal{O} \subset A.$$

The **interior** of a set A is the union of all the interior points of A, and is denoted by $\mathring{A}$.

Fact: a set A is open if and only if $A = \mathring{A}$.

A point $x \in X$ is a **point of closure** of a set A if every open nbhd of x intersects A.

The **closure** of A is the set of all the points of closure of A, and is denoted by $\overline{A}$.

Fact: a set A is closed if and only if $\overline{A} = A$ if and only if A is the intersection of all closed sets containing A.

A point $x \in X$ is a **cluster point** of a sequence $\{x_n\}$ in X if every open set containing x contains x_n for infinitely many n.

A sequence $\{x_n\}$ in X **converges to** $x \in X$ if for every open set $\mathcal{O}$ containing x there exists an integer $m(\mathcal{O})$ (depending on $\mathcal{O}$) such that $x_n \in \mathcal{O}$ for all $n \geq m(\mathcal{O})$.

A set B is **dense** in a set A if $A \subset \overline{B}$ (for example, if $A = (0,1)$ and $B = \mathbb{Q}$, we have $A \subset \overline{B}$); if $B \subset A$ and B is dense in A, then $\overline{A} = \overline{B}$; and B is dense in X if $\overline{B} = X$.

A topological space $\{X; \mathcal{U}\}$ is **separable** if it contains a countable dense subset.

For topological spaces $\{X; \mathcal{U}\}$ and $\{Y; \mathcal{V}\}$, a function $f : X \to Y$ is **continuous at** $x \in X$ if for each $V \in \mathcal{V}$ containing $f(x)$ there exists $\mathcal{O} \in \mathcal{U}$ containing x such that $f(\mathcal{O}) \subset V$.

A function $f : X \to Y$ is **continuous on** X if it is continuous at each $x \in X$.

Fact: $f : X \to Y$ is continuous if and only if for every $V \in \mathcal{V}$, the preimage $f^{-1}(V) = \{x \in X : f(x) \in V\}$ belongs to $\mathcal{U}$ (i.e., the preimage of any open set is open), if and only if the preimage of any closed set is closed.

A function from one topological space $\{X; \mathcal{U}\}$ to another $\{Y, \mathcal{V}\}$ is a **homeomorphism** if f is bijective and continuous, and f^{-1} is continuous.

4: Bases. A collection of open sets $\mathcal{B}$ is a **base** for the topology of $\{X; \mathcal{U}\}$ if for every $\mathcal{O} \in \mathcal{U}$ and every $x \in \mathcal{O}$ there exists $B \in \mathcal{B}$ such that $x \in B \subset \mathcal{O}$.

A collection $\mathcal{B}_x$ of open sets is a **base at** $x \in X$ if for every $\mathcal{O} \in \mathcal{U}$ containing x, there exists $B \in \mathcal{B}_x$ such that $x \in B \subset \mathcal{O}$.

Fact: $\mathcal{B}$ is a base for the topology of $\{X; \mathcal{U}\}$ if and only if it is a base for each $x \in X$.

Fact: given a base $\mathcal{B}$ for a topology $\mathcal{U}$ on X, a set $\mathcal{O}$ is open in X if and only if for each $x \in \mathcal{O}$ there is $B \in \mathcal{B}$ such that $x \in B \subset \mathcal{O}$.

Proposition 4.1. A collection $\mathcal{B}$ is a base for a topology $\mathcal{U}$ on a set X if and only if

(i) every $x \in X$ belongs to some $B \in \mathcal{B}$, and

(ii) for every $B_1, B_2 \in \mathcal{B}$, and every $x \in B_1 \cap B_2$, there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

See the Appendix of this Lecture Note for a proof.

Example. The standard topology on $\mathbb{R}^N$ has a base the collection $\mathcal{B}$ of open balls with centers in $\mathbb{Q}^N$ and radii in $\mathbb{Q}^+$ (the positive rational numbers).

The collection of open sets in $\mathbb{R}^N$ is uncountable while the set $\mathcal{B}$ is countable.

Appendix. Proof of Proposition 4.1. Suppose that a collection $\mathcal{B}$ is a base for a topology $\mathcal{U}$ on X.

Since $X \in \mathcal{U}$, we have for each $x \in X$ the existence of $B \in \mathcal{B}$ such that $x \in B \subset X$.

Since every element of $\mathcal{B}$ is an open set, we have for $B_1, B_2 \in \mathcal{B}$ that $B_1 \cap B_2$ is open and nonempty, so there is $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$.

Now for a given collection $\mathcal{B}$ satisfying (i) and (ii), we construct a collection of sets $\mathcal{U}$ in X that satisfies (a) $\emptyset, X \in \mathcal{U}$, (b) a union of any collection of sets in $\mathcal{U}$ is in $\mathcal{U}$, and (c) the intersection of finitely many elements of $\mathcal{U}$ is in $\mathcal{U}$.

Let $\mathcal{U}$ consist of $\emptyset$ and all nonempty subsets $\mathcal{O}$ of X such that for every $x \in \mathcal{O}$ there exists $B \in \mathcal{B}$ such that $x \in B \subset \mathcal{O}$.

Since for each $x \in X$, there is $B \in \mathcal{B}$ such that $x \in B$ by (i), we have $x \in B \subset X$ and so $X \in \mathcal{U}$.

For a collection $\{\mathcal{O}_\alpha : \alpha \in I\}$ of sets in $\mathcal{U}$, we have for each $\mathcal{O}_\alpha$ and any $x \in \mathcal{O}_\alpha$, the existence of $B_x \in \mathcal{B}$ such that $x \in B_x \subset \mathcal{O}_\alpha$.

It follows that for any $x \in \cup_{\alpha \in I} \mathcal{O}_\alpha$, that there is at least one choice of $\beta \in I$ such that $x \in \mathcal{O}_\beta$, and hence $x \in B_x \subset \mathcal{O}_\beta \subset \cup_{\alpha \in I} \mathcal{O}_\alpha$.

Thus $\cup_{\alpha \in I} \mathcal{O}_\alpha \in \mathcal{U}$.

Lastly, let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two elements of $\mathcal{U}$ with $\mathcal{O}_1 \cap \mathcal{O}_2 \neq \emptyset$.

For $x \in \mathcal{O}_1 \cap \mathcal{O}_2$, we have $x \in \mathcal{O}_1$ and $x \in \mathcal{O}_2$, so that by (i) there are $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset \mathcal{O}_1$ and $x \in B_2 \subset \mathcal{O}_2$.

This implies that $x \in B_1 \cap B_2$, so that by (ii) there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Since $B_1 \cap B_2 \subset \mathcal{O}_1 \cap \mathcal{O}_2$, we obtain $x \in B_3 \subset \mathcal{O}_1 \cap \mathcal{O}_2$.

By induction, the intersection of finitely many elements of $\mathcal{U}$ belongs to $\mathcal{U}$. $\square$