Math 541 Lecture #2
I.4: Bases, axioms of countability, and product topologies, Part II
I.9: Vector Spaces
I.10: Topological Vector Spaces
I.13: Metric Spaces, Part I

4: Axioms of Countability. A topological space $\{X;\mathcal{U}\}$ satisfies the **first axiom of countability** if each point $x \in X$ has a countable base.

A topological space $\{X;\mathcal{U}\}$ satisfies the **second axiom of countability** if there exists a countable base for its topology.

Proposition 4.2. Every topological space satisfying the second axiom of countability is separable (i.e., has a countable dense subset).

Proof. Let $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ be a countable base for the topology of $\{X;\mathcal{U}\}$.

For each $i \in \mathbb{N}$, select an element $x_i \in B_i$.

We show that the set $C = \{x_i : i \in \mathbb{N}\}$ is dense in X, i.e., that $(X =)\overline{X} = \overline{C}$.

For an arbitrary $x \in X$, let $\mathcal{O}_x$ be an open nbhd of x.

Since $\mathcal{B}$ is a base for the topology, there is $j \in \mathbb{N}$ such that $x \in B_j \subset \mathcal{O}_x$.

Since $x_j \in B_j$, we have that every open nbhd of x intersects C.

With x being arbitrary, we have that the closure of C is X. $\square$

4.1: Product Topologies. For two topological spaces $\{X_i,\mathcal{U}_i\}$, $i = 1, 2$, the **product topology** $\mathcal{U}_1 \times \mathcal{U}_2$ on the Cartesian product $X_1 \times X_2$ is constructed by considering the set $\mathcal{B}$ of all products $\mathcal{O}_1 \times \mathcal{O}_2$ for $\mathcal{O}_i \in \mathcal{U}_i$, $i = 1, 2$.

The sets $\mathcal{O}_1 \times \mathcal{O}_2$ are called **open rectangles** in product topology.

Fact: the collection $\mathcal{B}$ of open rectangles forms a base for a topology $\mathcal{U}_1 \times \mathcal{U}_2$ on $X_1 \times X_2$.

For the product topological space $\{X_1 \times X_2;\mathcal{U}_1 \times \mathcal{U}_2\}$, the projection maps

$$\pi_j : X_1 \times X_2 \to X_j,\ (x_1, x_2) \to x_j,\ j = 1, 2,$$

are continuous because $\pi_1^{-1}(\mathcal{O}) = \mathcal{O} \times X_2 \in \mathcal{U}_1 \times \mathcal{U}_2$ for all $\mathcal{O} \in \mathcal{U}_1$ (with a similar statement holding for π_2).

Fact: the topology $\mathcal{U}_1 \times \mathcal{U}_2$ is the weakest topology on $X_1 \times X_2$ for which the projection maps π_j are continuous.

9.1: Convex Sets (in Real Vector Spaces). A **convex combination** of two vectors x and y in a real vector space X is a vector of the form

$$tx + (1 - t)y,\ \text{where } t \in [0, 1].$$

As t varies over $[0, 1]$, the convex combination $tx + (1 - t)y$ traces a line segment whose extremities are x and y.

A convex combination of n vectors $x_1, \dots, x_n$ is a vector of the form

$$\sum_{i=1}^{n} \alpha_i x_i, \text{ where } \alpha_i \geq 0, \ \sum_{i=1}^{n} \alpha = 1.$$

A subset A of X is **convex** if for each pair of vectors $x, y \in A$, the elements $tx + (1-t)y$ belong to A for all $t \in [0, 1]$.

10: Topological Vector Spaces. A vector space X over $\mathbb{R}$ equipped with a topology $\mathcal{U}$ is a **topological vector space** if the vector space operations of addition and scalar multiplication,

$$+ : X \times X \to X, \ \cdot : \mathbb{R} \times X \to X,$$

are continuous with respect to the product topologies on $X \times X$ and $\mathbb{R} \times X$ respectively.

For a fixed $x_0 \in X$, the **translation by** x_0 on X is the function

$$T_{x_0}(x) = x + x_0, \ x \in X.$$

For a fixed $\lambda \in \mathbb{R} - \{0\}$, the **dilatation by** λ on X is the function

$$D_\lambda(x) = \lambda x, \ x \in X.$$

If $\{X, \mathcal{U}\}$ is a topological vector space over $\mathbb{R}$, then the maps T_{x_0} and D_λ are homeomorphisms from $\{X; \mathcal{U}\}$ to itself.

In particular, if $\mathcal{O}$ is open in X, then so is $T_{x_0}(\mathcal{O}) = x_0 + \mathcal{O}$ for all $x_0 \in X$, and the topology $\mathcal{U}$ on X is called **translation invariant**.

For a topological vector space $\{X; \mathcal{U}\}$, if $\mathcal{B}_\Theta$ is a base at the zero vector Θ, then for any fixed $x \in X$, the collection $\mathcal{B}_x = x + \mathcal{B}_\Theta$ is a base at the element x.

This means that the base $\mathcal{B}_\Theta$ at Θ determines the topology $\mathcal{U}$ on X.

If all the elements of $\mathcal{B}_\Theta$ are convex, the topology of $\{X; \mathcal{U}\}$ is called **locally convex**.

13: Metric Spaces. A **metric** on a nonempty set X is a function $d : X \times X \to \mathbb{R}$ that satisfies

(i) $d(x, y) \geq 0$ for all $(x, y) \in X \times X$ (nonnegativity),
(ii) $d(x, y) = 0$ if and only if $x = y$ (zero property),
(iii) $d(x, y) = d(y, x)$ for all $(x, y) \in X \times X$ (symmetric property), and
(iv) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (triangle inequality).

A **metric space** is a pair $\{X; d\}$ where d is a metric on X.

Define the **open ball centered at** $x \in X$ **with radius** $\rho > 0$ to be the set

$$B_\rho(x) = \{y \in X : d(x, y) < \rho\}.$$

Homework Problem 2A. Prove that the collection $\mathcal{B}$ of open balls $B_\rho(x)$ in a metric space satisfies the two conditions of Proposition 4.1.

Then the collection $\mathcal{B}$ generates a topology $\mathcal{U}$ on $\{X; d\}$, called the **metric topology**, with $\mathcal{B}$ as a base.

A set $\mathcal{O} \subset X$ is **open** in the metric topology if for every $x \in \mathcal{O}$ there is $B \in \mathcal{B}$ such that $x \in B \subset \mathcal{O}$.

A set $E \subset X$ is **closed** if $X - E$ is open.

A point $x \in X$ is a **point of closure** for a set E if $B_\epsilon(x) \cap E \neq \emptyset$ for all $\epsilon > 0$.

Fact: a set E is **closed** if and only if it coincides with the set all of its points of closure.

Homework Problem 2B. Prove that each singleton set $\{x\}$ is closed in the metric topology.

A point $x \in X$ is a **cluster point** of a sequence $\{x_n\}$ in X if for all $\epsilon > 0$ the open ball $B_\epsilon(x)$ contains infinitely many terms of $\{x_n\}$.

A sequence $\{x_n\}$ **converges to** a point $x \in X$ if for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $d(x, x_n) < \epsilon$ for all $n \geq n_\epsilon$.

A sequence $\{x_n\}$ is **Cauchy** in X if for all $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_\epsilon$.

A metric space $\{X; d\}$ is **complete** if every Cauchy sequence in X converges to an element of X.

For two metric spaces $\{X; d\}$ and $\{Y; \eta\}$, a function $f : X \to Y$ is **continuous at a point** $x \in X$ if for every ϵ there exists $\delta > 0$ (depending on x and ϵ) such that $\eta\{f(x), f(y)\} < \epsilon$ whenever $d(x, y) < \delta$.

A function $f : X \to Y$ is **continuous on** X if it is continuous at every $x \in X$.