Math 541 Lecture #4
II.1: Partitioning open subsets of $\mathbb{R}^N$
II.2: Limits of sets, characteristic functions, and σ-algebras

1. Partitioning open subsets of $\mathbb{R}^N$. How do we compute the volume of an open set in $\mathbb{R}^N$? If we could partition the open set into a countable union of "nice" subsets whose volumes are easy to compute, then the volume would be the sum of a series.

The "nice" subsets we will use are a particular type of cube whose volume is readily computable.

Let $q = (q_1, q_2, \ldots, q_N) \in \mathbb{Z}^N$, a N-tuple of integers.

For a fixed positive integer p and a fixed N-tuple of integers q, we define the $\frac{1}{2}$**-closed dyadic cube**

$$\mathcal{Q}_{p,q} = \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^N : \frac{q_i - 1}{2^p} < x_i \leq \frac{q_i}{2^p},\ i = 1, 2, \ldots, N \right\}.$$

All of $\mathbb{R}^N$ can be partitioned into $\frac{1}{2}$-closed dyadic cubes disjointly by slicing $\mathbb{R}^N$ by the hyperplanes $\{x_j = q_{l_j} 2^{-p}\}$ for $j = 1, 2, \ldots, N$ and q_{l_j} ranges over $\mathbb{Z}$.

Thus for each $p \in \mathbb{N}$ we have

$$\mathbb{R}^N = \bigcup_{q \in \mathbb{Z}^N} \mathcal{Q}_{p,q},$$

where

$$\mathcal{Q}_{p,q} \cap \mathcal{Q}_{p,q'} = \emptyset \text{ for all } q \neq q'.$$

Proposition 1.2. Each nonempty open subset E of $\mathbb{R}^N$ is the union of a countable collection of $\frac{1}{2}$-closed dyadic cubes with pairwise disjoint interiors.

Proof. (The proof in the Ed.1 of the book is jumbled).

Either finitely many (including 0) or countably many of the $\frac{1}{2}$-closed dyadic cubes $\mathcal{Q}_{1,q}$ are subsets of E, and we denote their union by

$$\mathcal{Q}_1 = \left\{ \bigcup \mathcal{Q}_{1,q} : \mathcal{Q}_{1,q} \subset E \right\} \subset E.$$

Either finitely many (including 0) or countably many of the $\frac{1}{2}$-closed dyadic cubes $\mathcal{Q}_{2,q}$ are subsets of $E - \mathcal{Q}_1$, and we denote their union by

$$\mathcal{Q}_2 = \left\{ \bigcup \mathcal{Q}_{2,q} : \mathcal{Q}_{2,q} \subset E - \mathcal{Q}_1 \right\} \subset E.$$

Continuing in this fashion determines for all $n \in \mathbb{N}$, the sets

$$\mathcal{Q}_n = \left\{ \bigcup \mathcal{Q}_{n,q} : \mathcal{Q}_{n,q} \subset E - \bigcup_{j=1}^{n-1} \mathcal{Q}_j \right\} \subset E.$$

The union of all the $\mathcal{Q}_n$ is a subset of E.

For each $x \in E$, there is a $\frac{1}{2}$-closed dyadic cube $\mathcal{Q}_{n,q}$ for a smallest $n \in \mathbb{N}$ and some $q \in \mathbb{Z}^N$ satisfying $x \in \mathcal{Q}_{n,q} \subset E$, and thus the union of the $\mathcal{Q}_n$ is all of E. $\square$

2. Limits of sets, characteristic functions, and σ-algebras. The **upper and lower limits of a sequence of subsets** $\{E_n\}$ in some set X are

$$E'' = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j = \limsup E_n, \quad E' = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} E_j = \liminf E_n.$$

The sequence $\{E_n\}$ **converges** (to a set) if $E'' = E'$, and we write

$$E'' = E' = \lim E_n.$$

A sequence $\{E_n\}$ is **monotone increasing** if $E_n \subset E_{n+1}$ for all n.

A monotone increasing $\{E_n\}$ has a limit because

$$E'' = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j = \bigcup_{n=1}^{\infty} E_n, \quad E' = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} E_j = \bigcup_{n=1}^{\infty} E_n,$$

are the same, where $\cup_{j=n}^{\infty} E_j$ is the same set for all n, and $\cap_{j=n}^{\infty} E_j = E_n$ for each n.

A sequence $\{E_n\}$ is **monotone decreasing** if $E_n \supset E_{n+1}$ for all n.

A monotone decreasing sequence $\{E_n\}$ has a limit because

$$E'' = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j = \bigcap_{n=1}^{\infty} E_n, \quad E' = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} E_j = \bigcap_{n=1}^{\infty} E_n,$$

are the same, where $\cup_{j=n}^{\infty} E_j = E_n$ for each n, and $\cap_{j=n}^{\infty} E_n$ is the same set for all n.

For the limit of $\{E_n\}$ to exist, it need not be monotone.

It is also possible for each $\{E_n\}$ to be infinite while the limit is the empty set: the sequence of closed intervals $E_n = [n, \infty)$ is monotone decreasing, while the limit is the empty set.

The **characteristic function** χ_E of set E in X is the real-valued function defined by

$$\chi_E = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Proposition. For a sequence of set $\{E_n\}$ we have

$$\chi_{\cup_{j=n}^{\infty} E_j} = \sup_{j \geq n} \chi_{E_j} \text{ and } \chi_{\cap_{j=n}^{\infty} E_j} = \inf_{j \geq n} \chi_{E_j}.$$

Proof. For $x \in X$, we have $\chi_{\cup_{j=n}^{\infty} E_j}(x) = 1$ if and only if $x \in \cup_{j=n}^{\infty} E_j$ if and only if $x \in E_j$ for some $j \geq n$ if and only if $\sup_{j \geq n} \chi_{E_j}(x) = 1$.

We also have $\chi_{\cap_{j=n}^{\infty} E_j}(x) = 1$ if and only if $x \in \cap_{j=n}^{\infty} E_j$ if and only if $x \in E_j$ for all $j \geq n$ if and only if $\inf_{j \geq n} \chi_{E_j}(x) = 1$. $\square$

A set E in a topological space $\{X;\mathcal{U}\}$ is **of type** $\mathcal{F}_\sigma$ if E is the union of a countable collection of closed subsets of X.

For example, the open interval $(-1, 1)$ is the union of the closed sets $[-1+1/n, 1-1/n]$.

A set E in a topological space $\{X;\mathcal{U}\}$ is **of type** $\mathcal{G}_\delta$ if E is the intersection of a countable collection of open subsets of X.

For example, the closed interval $[-1, 1]$ is the intersection of the open sets $(-1-1/n, 1+1/n)$.

A set E is **of type** $\mathcal{F}_{\sigma\delta}$ if it is the countable intersection of sets of type $\mathcal{F}_\sigma$.

A set E is **of type** $\mathcal{G}_{\delta\sigma}$ if it is the countable union of sets of type $\mathcal{G}_\delta$.

Sets of type $\mathcal{F}_{\sigma\delta\sigma}$ and $\mathcal{G}_{\delta\sigma\delta}$, etc., are similarly defined.

A collection $\mathcal{A}$ of subsets of a set X is an **algebra of sets** if it contains the empty set, if the union of any two elements in $\mathcal{A}$ is in $\mathcal{A}$, and if the complement of any element of $\mathcal{A}$ is in $\mathcal{A}$.

It follows that an algebra of sets $\mathcal{A}$ in X contains X and the union and intersection of finitely many elements of $\mathcal{A}$.

A collection of subsets $\mathcal{A}$ of a set X is a σ**-algebra** if $\mathcal{A}$ is an algebra for which the union of a countable collection of elements of $\mathcal{A}$ is in $\mathcal{A}$.

The power set $\mathcal{A} = 2^X = \mathcal{P}(X)$ is a σ-algebra known as the **discrete** σ-algebra.

The collection $\mathcal{A} = \{\emptyset, X\}$ is a σ-algebra known as the **trivial** σ-algebra.

Proposition 2.1. For a given collection $\mathcal{O}$ of subsets of X, there exists a smallest σ-algebra $\mathcal{A}_\mathcal{O}$ that contains $\mathcal{O}$.

Proof. Let $\mathcal{F}$ be the collection of all σ-algebras that contain $\mathcal{O}$.

The collection $\mathcal{F}$ is nonempty because it contains the discrete σ-algebra $\mathcal{P}(X)$.

Define
$$\mathcal{A}_\mathcal{O} = \bigcap \{\mathcal{A} : \mathcal{A} \in \mathcal{F}\}.$$

Two sets $A_1, A_2 \in \mathcal{A}_\mathcal{O}$ belong to every σ-algebra in $\mathcal{F}$, so the union $A_1 \cup A_2$ belongs to every σ-algebra in $\mathcal{F}$, and hence $A_1 \cup A_2$ in $\mathcal{A}_\mathcal{O}$.

A set $A \in \mathcal{A}_\mathcal{O}$ belongs to every σ-algebra in $\mathcal{F}$, so the complement A^c belongs to every σ-algebra in $\mathcal{F}$, and hence A^c is in $\mathcal{A}_\mathcal{O}$.

Thus $\mathcal{A}_\mathcal{O}$ is an algebra of sets.

Each countable collection $\{A_n\}$ in $\mathcal{A}_\mathcal{O}$ belongs to every element of $\mathcal{F}$ and so the union $\cup_{n=1}^\infty A_n$ belongs to every element of $\mathcal{F}$ and hence $\cup_{n=1}^\infty A_n$ belongs to $\mathcal{A}_\mathcal{O}$.

Thus $\mathcal{A}_\mathcal{O}$ is a σ-algebra.

It is the smallest containing $\mathcal{O}$ because if $\mathcal{A}'$ is another σ-algebra containing $\mathcal{O}$, then $\mathcal{A}' \in \mathcal{F}$, so that $\mathcal{A}_\mathcal{O} \subset \mathcal{A}'$. $\square$

The **Borel** σ**-algebra** $\mathcal{B}$ is the smallest σ-algebra containing all of the open subsets $\mathcal{U}$ of X; it contains all of the closed subsets of X, and all of the $\mathcal{F}_\sigma$, $\mathcal{G}_\delta$, $\mathcal{F}_{\sigma\delta}$, $\mathcal{G}_{\delta\sigma}$, etc., type sets.