Math 541 Lecture #5
II.3: Measures, Part I

3. Measures. The set of **extended real numbers** is denoted by

$$\mathbb{R}^* = \{-\infty\} \cup \mathbb{R} \cup \{\infty\},$$

where for any $c \in \mathbb{R}$ we define $\pm\infty \pm c = \pm\infty$ and $(\pm\infty)c = (\pm\infty)\text{sign}(c)$.

We further set $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$, but leave $\infty - \infty$ undefined.

Let $\mathcal{A}$ be a σ-algebra of sets in X.

A function $\mu : \mathcal{A} \to \mathbb{R}^*$ is **countably additive** if for any finite or countable collection of pairwise disjoint sets $\{E_n\}$ in $\mathcal{A}$, we have

$$\mu\left(\bigcup E_n\right) = \sum \mu(E_n).$$

A function $\mu : \mathcal{A} \to \mathbb{R}^*$ is **countably subadditive** if for any finite or countable collection $\{E_n\}$ of sets in $\mathcal{A}$ we have

$$\mu\left(\bigcup E_n\right) \leq \sum \mu(E_n).$$

A **measure** on a σ-algebra $\mathcal{A}$ is a countably additive function $\mu : \mathcal{A} \to \mathbb{R}^*$ such that $\mu(A) \geq 0$ for all $A \in \mathcal{A}$, and $\mu(A) < \infty$ for some $A \in \mathcal{A}$.

A **measure space** is a triple $\{X, \mathcal{A}, \mu\}$ where $\mathcal{A}$ is a σ-algebra in X and $\mu : \mathcal{A} \to \mathbb{R}^*$ is a measure.

Proposition 3.1. For a σ-algebra $\mathcal{A}$ in a set X, suppose $\mu : \mathcal{A} \to \mathbb{R}^*$ is a measure.

(i) For any $A, B \in \mathcal{A}$ with $A \subset B$, we have

$$\mu(A) \leq \mu(B) \text{ (known as monotonicity), and,}$$
$$\mu(B - A) = \mu(B) - \mu(A) \text{ if } \mu(A) < \infty.$$

(ii) $\mu(\emptyset) = 0$.

(iii) For any $A, B \in \mathcal{A}$, we have if $\mu(A \cap B) < \infty$ then

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

(iv) μ is countably subadditive.

(v) For a countable collection $\{E_n\}$ in $\mathcal{A}$ we have

$$\liminf \mu(E_n) \geq \mu(\liminf E_n).$$

(vi) For a countable collection $\{E_n\}$ in $\mathcal{A}$, if $\mu(\cup E_n) < \infty$, then

$$\limsup \mu(E_n) \leq \mu(\limsup E_n).$$

Proof. (i) Since $B = A \cup (B - A)$ is a disjoint union of sets in $\mathcal{A}$, the countable additivity of μ gives

$$\mu(B) = \mu(A) + \mu(B - A).$$

Since $\mu(B - A) \geq 0$, we obtain $\mu(A) \leq \mu(B)$.

If $\mu(A) < \infty$, we avoid the undefined $\infty - \infty$ to get $\mu(B - A) = \mu(B) - \mu(A)$.

(ii) There is a set $E \in \mathcal{A}$ for which $\mu(E) < \infty$.

Applying part (i) with $A = E$ and $B = E$ gives

$$\mu(\emptyset) = \mu(E - E) = \mu(E) - \mu(E) = 0.$$

(iii) Since $A \cup B = A \cup (B - (A \cap B))$ is a disjoint union of sets in $\mathcal{A}$, we have by countable additivity of μ that

$$\mu(A \cup B) = \mu(A) + \mu(B - (A \cap B)).$$

Since $A \cap B \subset B$ and $\mu(A \cap B) < \infty$, we have by part (i) that $\mu(B - (A \cap B)) = \mu(B) - \mu(A \cap B)$, and so

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

(iv) Let $\{E_n\}$ be a finite or countable collection of sets in $\mathcal{A}$.

If we set $B_1 = E_1$, and inductively set

$$B_n = E_n - \bigcup_{j=1}^{n-1} E_j, \ n \geq 2,$$

then $\{B_n\}$ is a finite or countable collection of pairwise disjoint sets in $\mathcal{A}$ whose union is the same as the union of $\{E_n\}$.

Thus by countable additivity of μ and part (i) we have

$$\mu\left(\bigcup E_n\right) = \mu\left(\bigcup B_n\right) = \sum \mu(B_n) = \sum \mu\left(E_n - \bigcup_{j=1}^{n-1} E_j\right) \leq \sum \mu(E_n).$$

(v) Let $\{E_n\}$ be a countable collection in $\mathcal{A}$.

Recall the definition of the lim inf applied to the sequence $\{\mu(E_n)\}$:

$$\liminf \mu(E_n) = \sup_{n \geq 1} \inf_{j \geq n} \mu(E_j) = \lim_{n \to \infty} \inf_{j \geq n} \mu(E_j).$$

If $\liminf \mu(E_n) = \infty$, then trivially we have $\liminf \mu(E_n) \geq \mu(\liminf E_n)$.

So suppose that $\liminf \mu(E_n) < \infty$.

Then for every $n \in \mathbb{N}$ we have infinitely many $k \geq n$ such that $\mu(E_k) < \infty$.

Set $D_n = \cap_{j \geq n} E_j \in \mathcal{A}$.

Then

$$\liminf E_n = \bigcup_{n=1}^{\infty} \bigcap_{j \geq n} E_j = \bigcup_{n=1}^{\infty} D_n \in \mathcal{A}.$$

The collection $\{D_n\}$ is monotone increasing, i.e., $D_n \subset D_{n+1}$ for all n.

We turn the union of the D_n's into a disjoint union by

$$\bigcup_{n=1}^{\infty} D_n = D_1 \bigcup \left(\bigcup_{n=1}^{\infty} (D_{n+1} - D_n) \right).$$

Since for each n there are infinitely many $k \geq n$ such that $\mu(E_k) < \infty$ and since $D_n = \cap_{j \geq n} E_n \subset E_k$ for all $k \geq n$, we have that $\mu(D_n) < \infty$ for all n.

Thus by the countable additivity of μ and part (i) we have

$$\begin{aligned}
\mu(\liminf E_n) = \mu\left(\bigcup_{n=1}^{\infty} D_n \right) &= \mu(D_1) + \sum_{n=1}^{\infty} \mu(D_{n+1} - D_n) \\
&= \mu(D_1) + \sum_{n=1}^{\infty} \big(\mu(D_{n+1}) - \mu(D_n) \big) = \lim_{n \to \infty} \mu(D_n)
\end{aligned}$$

Since $D_n = \cap_{k \geq n} E_k \subset E_j$ for all $j \geq n$, we have by monotonicity of μ that

$$\mu(D_n) \leq \mu(E_j) \text{ for all } j \geq n.$$

Thus for each $n \in \mathbb{N}$ we have

$$\mu(D_n) \leq \inf_{j \geq n} \mu(E_n),$$

and we obtain

$$\lim_{n \to \infty} \mu(D_n) = \sup_{n \geq 1} \mu(D_n) \leq \sup_{n \geq 1} \inf_{j \geq n} \mu(E_j) = \liminf_{n \to \infty} \mu(E_n).$$

Hence

$$\mu(\liminf E_n) \leq \liminf \mu(E_n).$$

(vi) The proof of this is in the Appendix of this Lecture Note.

Homework Problem 5A. For a measure μ on a σ-algebra $\mathcal{A}$, prove that if $\{E_n\}$ in $\mathcal{A}$ is monotone increasing and $E = \cup E_n$, then $\mu(E_n) \to \mu(E)$ as $n \to \infty$.

Homework Problem 5B. For a measure μ on a σ-algebra $\mathcal{A}$, prove that if $\{E_n\}$ in $\mathcal{A}$ is monotone decreasing, there exists $k \in \mathbb{N}$ such that $\mu(E_k) < \infty$, and $E = \cap E_n$, then $\mu(E_n) \to \mu(E)$ as $n \to \infty$; show that this is false if there is no $k \in \mathbb{N}$ such that $\mu(E_k) < \infty$.

Appendix Proof of part (vi) of Proposition 3.1. For a countable collection $\{E_n\}$, if we set $A_n = \cup_{j \geq n} E_j \in \mathcal{A}$, $n = 1, 2, 3, \ldots$, then the sequence of sets

$$C_n = A_1 - A_n = \bigcup_{k=1}^{\infty} E_k - \bigcup_{j \geq n} E_j \in \mathcal{A},\ n = 1, 2, 3, \ldots,$$

is monotone increasing.

Since $A_n \subset A_1$ and by hypothesis $\mu(A_1) < \infty$, we have by part (i) that $\mu(C_n) = \mu(A_1 - A_n) = \mu(A_1) - \mu(A_n) < \infty$.

For

$$A = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{j \geq n} E_j = \limsup E_n \in \mathcal{A},$$

we have

$$\begin{aligned} A_1 - A = A_1 - \bigcap_{n=1}^{\infty} A_n &= A_1 \cap \left(\bigcap_{n=1}^{\infty} A_n \right)^c = A_1 \cap \left(\bigcup_{n=1}^{\infty} A_n^c \right) \\ &= \bigcup_{n=1}^{\infty} (A_n^c \cap A_1) = \bigcup_{n=1}^{\infty} (A_1 - A_n) = \bigcup_{n=1}^{\infty} C_n. \end{aligned}$$

Since $A \subset A_1$ and $\mu(A_1) < \infty$, we have by part (i) that $\mu(A_1) - \mu(A) = \mu(A_1 - A)$.

By setting $D_1 = C_1$ and $D_{n+1} = C_{n+1} - C_n$ for $n = 1, 2, 3, \ldots$ we have $\cup C_n = \cup D_n$ where $D_i \cap D_j = \emptyset$ for all $i \neq j$, so that by countable additivity, we have

$$\mu(\cup C_n) = \mu(\cup D_n) = \sum \mu(D_n) = \mu(C_1) + \sum_{n=1}^{\infty} \big(\mu(C_{n+1}) - \mu(C_n) \big) = \lim_{n \to \infty} \mu(C_n),$$

where we have used $C_n \subset C_{n+1}$ (from increasing monotonicity of $\{C_n\}$) and part (i).

Thus we obtain

$$\mu(A_1) - \mu(A) = \lim_{n \to \infty} \mu(C_n) = \lim_{n \to \infty} \big(\mu(A_1) - \mu(A_n) \big) = \mu(A_1) - \lim_{n \to \infty} \mu(A_n).$$

By hypothesis, $\mu(A_1) < \infty$, so that $\mu(A) = \lim_{n \to \infty} \mu(A_n)$.

Since $A_n \supset E_j$ for all $j \geq n$, we have $\mu(A_n) \geq \mu(E_j)$ for all $j \geq n$ by monotonicity.

Thus $\mu(A_n) \geq \sup_{j \geq n} \mu(E_j)$ for each $n \in \mathbb{N}$, and hence

$$\mu(\limsup E_n) = \mu(A) = \lim_{n \to \infty} \mu(A_n) \geq \inf_{n \geq 1} \sup_{j \geq n} \mu(E_j) = \limsup_{n \to \infty} \mu(E_n),$$

the last equality being the definition of lim sup. $\square$