Math 541 Lecture #6
II.3: Measures, Part II
II.4: Outer measures and sequential coverings, Part I

3.1: Finite, σ-finite, and complete measures. Let μ be a measure on a σ-algebra $\mathcal{A}$ in a set X.

The measure μ is **finite** if $\mu(X) < \infty$.

The measure μ is **σ-finite** if there exists a countable collection $\{E_n\}$ in $\mathcal{A}$ such that

$$X = \bigcup_{n=1}^{\infty} E_n \text{ and } \mu(E_n) < \infty \ \forall n.$$

A measure space $\{X, \mathcal{A}, \mu\}$ is **complete** if for each $A \in \mathcal{A}$ with $\mu(A) = 0$, every subset $E \subset A$ is in $\mathcal{A}$.

It follows from the monotonicity of a measure, that if $\{X, \mathcal{A}, \mu\}$ is complete, then for each $A \in \mathcal{A}$ with $\mu(A) = 0$, we have for every $E \subset A$ that $\mu(E) = 0$.

3.2: Some Examples. (a) For any nonempty set X and $\mathcal{A} = \{\emptyset, X\}$, the trivial σ-algebra, the function $\mu : \mathcal{A} \to \mathbb{R}^*$ defined by

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset, \\ \infty & \text{if } E = X, \end{cases}$$

is a measure.

(b) For a nonempty set X and $\mathcal{A} = 2^X = \mathcal{P}(X)$ the discrete σ-algebra, the function $\mu : \mathcal{A} \to \mathbb{R}^*$ defined by setting $\mu(E)$ equal to the number of elements of E if E is a finite set, and setting $\mu(E) = \infty$ if E is not a finite set, is a measure, called the counting measure.

(c) Let $X = \{x_n\}$ be a countably infinite set, and $\{\alpha_n\}$ a sequence of nonnegative real numbers.

The function

$$\mu(E) = \sum \{\alpha_n : x_n \in E\}$$

is a σ-finite measure on the discrete σ-algebra $\mathcal{A} = 2^X$.

This measure is finite if $\sum \alpha_n < \infty$.

(d) For an infinite set X (possibly uncountable), and the discrete σ-algebra $\mathcal{A} = 2^X$, the function $\mu : \mathcal{A} \to \mathbb{R}^*$ defined by $\mu(E) = 0$ if E is countable (including finite), and $\mu(E) = \infty$ otherwise, is a measure.

(e) Let $X = \mathbb{R}^n$ and $\mathcal{A} = \mathcal{P}(\mathbb{R}^n)$.

For a fixed $x \in \mathbb{R}^n$ we define $\mu : \mathcal{A} \to \mathbb{R}^*$ by

$$\mu(E) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E, \end{cases}$$

is a finite measure, known as the **Dirac delta** measure δ_x in $\mathbb{R}^N$ concentrated at x.

Homework problem 6A. Give an example of a measure μ for which $\mu(B - A) = \mu(B) - \mu(A)$ fails when $\mu(A) = \infty$.

Proposition A. If $\{\mu_\alpha : \alpha \in I\}$ is a finite or countable infinite collection of measures on the same σ-algebra $\mathcal{A}$, then $\sum \mu_\alpha$ is a measure on $\mathcal{A}$.

Homework problem 6B. Give a proof of Proposition A.

4. Outer Measures. An extended real-valued set function μ_e on X is an **outer measure** if

(i) μ_e is defined for every element of $\mathcal{P}(X)$,
(ii) μ_e is nonnegative and $\mu_e(\emptyset) = 0$,
(iii) μ_e is monotone, i.e., if $A \subset B$, then $\mu_e(A) \leq \mu_e(B)$, and
(iv) μ_e is countably subadditive, i.e., for $\{A_n\} \in \mathcal{P}(X)$, there holds

$$\mu_e(\cup A_n) \leq \sum \mu_e(A_n).$$

A collection $\mathcal{Q}$ of subsets of a set X is a **sequential covering** for X if

(i) $\emptyset \in \mathcal{Q}$, and
(ii) for every $E \subset X$ there is a countable collection $\{Q_n\}$ in $\mathcal{Q}$ such that

$$E \subset \bigcup_{n=1}^{\infty} Q_n.$$

Example. A sequential covering of $\mathbb{R}^n$ is the collection of all closed cubes.

We describe a general procedure by which an outer measure is constructed from a sequential covering $\mathcal{Q}$ of set X and an arbitrary nonnegative function $\lambda : \mathcal{Q} \to \mathbb{R}^*$ satisfying $\lambda(\emptyset) = 0$.

For each $E \in \mathcal{P}(X)$, we define $\mu_e : \mathcal{P}(X) \to \mathbb{R}^*$ by

$$\mu_e(E) = \inf \left\{ \sum_{n=1}^{\infty} \lambda(Q_n) : Q_n \in \mathcal{Q}, E \subset \bigcup_{n=1}^{\infty} Q_n \right\}.$$

By the definition of inf, if $\mu_e(E) < \infty$, then for every $\epsilon > 0$ there is a countable collection $\{Q_{n,\epsilon}\}$ of elements in $\mathcal{Q}$ such that

$$E \subset \bigcup_{n=1}^{\infty} Q_{n,\epsilon} \text{ and } \bigcup_{n=1}^{\infty} \lambda(Q_{n,\epsilon}) \leq \mu_e(E) + \epsilon.$$

Proposition 4.1. The function μ_e is an outer measure.

Proof. We have four properties to verify.

(i) $\mu_e(E)$ is defined on every element of $\mathcal{P}(X)$: this follows because the infimum is defined for each $E \in \mathcal{P}(X)$.

(ii) $\mu_e(E) \geq 0$ for all $E \in \mathcal{P}(X)$ and $\mu(\emptyset) = 0$: these follows because λ is nonnegative, and $\lambda(\emptyset) = 0$ and the constant sequence $\{\emptyset\}$ is a sequential covering of $\emptyset$.

(iii) μ_e is monotone, i.e., $A \subset B$ implies $\mu_e(A) \leq \mu_e(B)$: this follows because every sequential cover of B is a sequential cover of A, but not every sequential of A is a sequential cover for B, so that the infimum for $\mu_e(A)$ is smaller or equal to that for $\mu_e(B)$.

(iv) μ_e is countably subadditive.

We assume for a countable collection $\{E_n\}$ of elements of $\mathcal{P}(X)$ that $\mu_e(E_n) < \infty$ for all n (for otherwise countable subadditivity follows trivially).

Fix $\epsilon > 0$.

For each $n \in \mathbb{N}$, there is a countable collection $\{Q_{j,n}\}$ in $\mathcal{Q}$ such that

$$E_n \subset \bigcup_{j=1}^{\infty} Q_{j,n} \text{ and } \sum_{j=1}^{n} \lambda(Q_{j,n}) \leq \mu_e(E_n) + \frac{\epsilon}{2^n}.$$

The doubly-indexed collection $\{Q_{j,n}\}$ is a countable collection that covers the union of the E_n so that

$$\mu_e\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \lambda(Q_{n,j}) \leq \sum_{n=1}^{\infty} \mu_e(E_n) + \epsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \mu_e(E_n) + \epsilon.$$

Since this holds for any $\epsilon > 0$ we obtain the countable subadditivity of μ_e. $\square$

The outer measure μ_e generated by the sequential covering $\mathcal{Q}$ and the nonnegative function λ may not coincide with λ on elements of $\mathbb{Q}$.

By the construction of μ_e we have for all $Q \in \mathcal{Q}$ that

$$\mu_e(Q) \leq \lambda(Q),$$

and strict inequality may occur for some Q. [We will see some examples of this soon.]