Math 541 Lecture #7
II.4: Outer measures and sequential coverings, Part II
II.5: The Hausdorff outer measure in $\mathbb{R}^N$, Part I

4.1: The Lebesgue outer measure in $\mathbb{R}^N$. Let $\mathcal{Q}$ denote the collection of all $\frac{1}{2}$-closed dyadic cubes in $\mathbb{R}^N$.

These cubes are of the form

$$Q_{p,q} = \left\{ x \in \mathbb{R}^N : \frac{q_i - 1}{2^p} < x_i \leq \frac{q_i}{2^p} \right\}.$$

For such a cube Q, we denote by $\operatorname{diam}(Q)$ the length of a diameter of the cube:

$$\operatorname{diam}(Q) = \sup_{x,y\in Q} |x-y| = \sqrt{\left(\frac{1}{2^p}\right)^2 + \cdots + \left(\frac{1}{2^p}\right)^2} = \frac{\sqrt{N}}{2^p}.$$

For the function λ defined on $\mathbb{Q}$ we take

$$\lambda(Q) = \left(\frac{\operatorname{diam}(Q)}{\sqrt{N}}\right)^N = \left(\frac{\sqrt{N}}{2^p\sqrt{N}}\right)^N = \left(\frac{1}{2^p}\right)^N,$$

which is the volume of Q.

Since every subset of $\mathbb{R}^N$ can be contained in some open subset, and since $\mathcal{Q}$ is a sequential covering of $\mathbb{R}^N$, we can define for every $E \in \mathcal{P}(\mathbb{R}^N)$,

$$\mu_e(E) = \inf \left\{ \sum \left(\frac{\operatorname{diam}(Q_n)}{\sqrt{N}}\right)^N : E \subset \bigcup Q_n, \{Q_n\} \subset \mathcal{Q} \right\},$$

which is the **Lebesgue outer measure** of $\mathbb{R}^N$.

4.2: The Lebesgue-Stieltjes outer measure in $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing and right-continuous, i.e., $x \leq y$ implies $f(x) \leq f(y)$, and $\lim_{x\to a^+} f(x) = f(a)$ for all $a \in \mathbb{R}$ (where the right-sided limit exists and is equal to $\inf_{x>a} f(x)$ by the monotone increasing of f).

Let $\mathcal{Q}$ be the collection of open subintervals of $\mathbb{R}$, which is a sequential covering of $\mathbb{R}$ (see Proposition 1 in II.1).

Define $\lambda_f : \mathcal{Q} \to \mathbb{R}$ by

$$\lambda_f\big((a,b)\big) = f(b) - f(a).$$

For $E \in \mathcal{P}(\mathbb{R})$, the outer measure generated by $\mathcal{Q}$ and λ is

$$\mu_{f,e}(E) = \inf \left\{ \sum \lambda_f(Q_n) : E \subset \bigcup Q_n, \{Q_n\} \subset \mathcal{Q} \right\},$$

and is called the **Lebesgue-Stieltjes outer measure generated by** f.

Proposition. $\mu_{f,e}\big((a,b]\big) = f(b) - f(a)$.

Proof. Recall for a nonempty bounded below set of real numbers S and a lower bound α of S, that $\alpha = \inf S$ if and only if for all $\epsilon > 0$ there exists $s \in S$ such that $s < \alpha + \epsilon$.

Thus, if we can show that $f(b) - f(a)$ is a lower bound, and for every $\epsilon > 0$ we can exhibit the existence of a sequential covering $\{Q_n\}$ of $(a,b]$ such that

$$\sum \lambda_f(Q_n) < f(b) - f(a) + \epsilon,$$

then we have implied that $\mu_{f,e}\big((a,b]\big) = f(b) - f(a)$.

To show that $f(b) - f(a)$ is a lower bound, suppose that it is not.

Then there is $\{Q_n\} \subset \mathcal{Q}$ such that $(a,b] \subset \cup Q_n$ and

$$\sum \lambda_f(Q_n) < f(b) - f(a).$$

Since $(a,b] \subset \cup Q_n$, there is an open interval (c,d) such that $(a,b] \subset (c,d) \subset \cup Q_n$.

Since $(c,d) \subset \cup Q_n = \cup(a_n,b_n)$, the monotonicity of f and our assumption imply

$$f(d) - f(c) = \lambda_f\big((c,d)\big) \leq \sum \lambda_f\big((a_n,b_n)\big) = \sum \lambda_f(Q_n) < f(b) - f(a).$$

But $c \leq a < b < d$ which implies that $f(b) - f(a) \leq f(d) - f(c)$, a contradiction.

For $\epsilon > 0$ there is by the right continuity of f at b the existence of $\delta > 0$ such that for all $x \in (b, b+\delta)$ we have $f(x) - f(b) < \epsilon$.

Select $b_\epsilon \in (b, b+\delta)$.

The sequential covering of $(a,b]$ consisting of just the one open interval (a,b_ϵ) satisfies

$$\lambda_f\big((a,b_\epsilon)\big) = f(b_\epsilon) - f(a) < f(b) - f(a) + \epsilon.$$

Therefore we have that $\mu_{f,e}\big((a,b]\big) = f(b) - f(a)$. □

Does $\mu_{f,e}\big((a,b)\big) = \lambda_f\big((a,b)\big)$ for all $(a,b) \in \mathcal{Q}$?

Proposition. If f is discontinuous at b but continuous on $[a-\eta, b)$ for a small $\eta > 0$, then $\mu_{f,e}\big((a,b)\big) < f(b) - f(a)$.

Proof. Suppose f is discontinuous at b, i.e., there is a jump discontinuity of f at b, while f is continuous on $[a-\eta,b)$ for some small $\eta > 0$.

Set

$$\epsilon = \lim_{x\to b^+} f(x) - \lim_{x\to b^-} f(x) = f(b) - \lim_{x\to b^-} f(x) > 0.$$

We will choose a sequential covering $\{Q_n\}$ of (a,b) as follows.

We want the endpoints of the open intervals $Q_n = (a_n,b_n)$ to satisfy

1. $\{a_n\}$ is strictly increasing with $a_1 = a$,
2. $\{b_n\}$ is strictly increasing with $b_n < b$ for all n,
3. $a_{n+1} < b_n$ for all n.

Since $b_n \in [a-\eta, b)$, we can choose a_{n+1} close enough to b_n by the continuity of f at b_n so that

$$f(b_n) - f(a_{n+1}) \le \frac{\epsilon}{2^{n+1}}.$$

Then we have for each positive integer m that

$$\begin{aligned}\sum_{n=1}^{m} \big(f(b_n) - f(a_n)\big) &= -f(a_1) + (f(b_1) - f(a_2)) + (f(b_2) - f(a_3)) + \cdots \\ &\quad + (f(b_{m-1}) - f(a_m)) + f(b_m) \\ &= f(b_m) - f(a_1) + \sum_{n=1}^{m-1} \frac{\epsilon}{2^{n+1}} \le \lim_{x\to b^-} f(x) - f(a) + \frac{\epsilon}{2},\end{aligned}$$

where we have used the monotonicity of f.

This upper bound on the partial sums implies that

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_f(Q_n) &< \lim_{x\to b^-} f(x) - f(a) + \epsilon \\ &= \lim_{x\to b^-} f(x) - f(a) + f(b) - \lim_{x\to b^-} f(x) \\ &= f(b) - f(a).\end{aligned}$$

Therefore $\mu_{f,e}\big((a,b)\big) < f(b) - f(a) = \lambda_f\big((a,b)\big)$. □

Homework Problem 7A. For $f(x) = e^x$, find $\mu_{f,e}\big((a,b)\big)$.

5. Hausdorff Outer Measure in $\mathbb{R}^N$. For $\epsilon > 0$, let $\mathcal{E}_\epsilon$ be the sequential covering of $\mathbb{R}^N$ consisting of all subsets E of $\mathbb{R}^N$ for which $\mathrm{diam}(E) < \epsilon$.

For fixed $\alpha > 0$ and $E \in \mathcal{E}_\epsilon$ define $\lambda(E) = (\mathrm{diam}(E))^\alpha$; and set $\lambda(\emptyset) = 0$.

The nonnegative function λ on $\mathcal{E}_\epsilon$ generates an outer measure

$$\mathcal{H}_{\alpha,\epsilon}(E) = \inf\left\{\sum \big(\mathrm{diam}(E_n)\big)^\alpha : E \subset \bigcup E_n, \{E_n\} \in \mathcal{E}_\epsilon\right\}.$$

For this outer measure, we have $\mathcal{H}_{\alpha,\epsilon}(E) < \lambda(E)$ when $\alpha > 2$ and E is a square of unit edge in $\mathbb{R}^2$ because

$$\lambda(E) = \big(\sqrt{1+1}\big)^\alpha = \big(\sqrt{2}\big)^\alpha,$$

while for all $\alpha > 2$ there holds

$$\mathcal{H}_{\alpha,\epsilon}(E) = 0.$$

Homework problem 7B. Prove that $\mathcal{H}_{\alpha,\epsilon}(E) = 0$ for all $\alpha > 2$.

If $\epsilon' < \epsilon$ then $\mathcal{H}_{\alpha,\epsilon} \le \mathcal{H}_{\alpha,\epsilon'}$ because every sequential covering by sets in $\mathcal{E}_{\epsilon'}$ is a sequential covering by sets in $\mathcal{E}_\epsilon$, but not every sequential covering by sets in $\mathcal{E}_\epsilon$ is a sequential covering by sets in $\mathcal{E}_{\epsilon'}$. This implies that

$$\mathcal{H}_\alpha(E) = \sup_{\epsilon>0} \mathcal{H}_{\alpha,\epsilon}(E) = \lim_{\epsilon\to 0} \mathcal{H}_{\alpha,\epsilon}(E)$$

exists in $\mathbb{R}^*$ and defines a nonnegative function $\mathcal{H}_\alpha$ on $\mathcal{P}(\mathbb{R}^N)$.