Math 541 Lecture #8
II.5: The Hausdorff outer measure in $\mathbb{R}^N$, Part II

Recall that

$$\mathcal{H}_{\alpha,\epsilon}(E) = \inf\left\{\sum \big(\mathrm{diam}(E_n)\big)^\alpha : E \subset \bigcup E_n, \{E_n\} \in \mathcal{E}_\epsilon\right\}$$

is an outer measure, and that

$$\mathcal{H}_\alpha(E) = \sup_{\epsilon>0} \mathcal{H}_{\alpha,\epsilon}(E) = \lim_{\epsilon\to 0} \mathcal{H}_{\alpha,\epsilon}(E)$$

is a nonnegative function on $\mathcal{P}(\mathbb{R}^N)$

Proposition 5.1. The function $\mathcal{H}_\alpha$ is an outer measure on $\mathbb{R}^N$. Moreover, for $\alpha < \beta$, there holds

(i) if $\mathcal{H}_\alpha(E) < \infty$, then $\mathcal{H}_\beta(E) = 0$, and
(ii) if $\mathcal{H}_\beta(E) > 0$, then $\mathcal{H}_\alpha(E) = \infty$.

Lastly, for every $E \in \mathcal{P}(\mathbb{R}^N)$ there holds

$$\frac{\mathcal{H}_N(E)}{N^{N/2}} \le \mu_e(E) \le \frac{\kappa_N \mathcal{H}_N(E)}{2^N}$$

where κ_N is the volume of the unit ball in $\mathbb{R}^N$ and μ_e is the Lebesgue outer measure in $\mathbb{R}^N$. In particular when $N = 1$, the Lebesgue outer measure coincides with the Hausdorff outer measure $\mathcal{H}_1$. (See Appendix in this lecture note for definition of κ_N.)

Proof. That $\mathcal{H}_\alpha$ is an outer measures follows from mimicking the proof that μ_e is an outer measure (Proposition 4.1).

(i) Let $\{E_n\} \subset \mathcal{E}_\epsilon$ be a sequential covering of $E \in \mathcal{P}(\mathbb{R}^N)$.

Since $\beta > \alpha$ we have for $0 \le a < \epsilon$ that $a^{\beta-\alpha} < \epsilon^{\beta-\alpha}$, so that $a^\beta < \epsilon^{\beta-\alpha} a^\alpha$; thus

$$\mathcal{H}_{\beta,\epsilon}(E) \le \sum_{n=1}^\infty \big(\mathrm{diam}(E_n)\big)^\beta \le \epsilon^{\beta-\alpha} \sum_{n=1}^\infty \big(\mathrm{diam}(E_n)\big)^\alpha.$$

This implies that

$$\mathcal{H}_{\beta,\epsilon}(E) \le \epsilon^{\beta-\alpha} \mathcal{H}_{\alpha,\epsilon}(E).$$

If $\mathcal{H}_\alpha(E) < \infty$, then $\mathcal{H}_\alpha(E) = \sup_{\epsilon>0} \mathcal{H}_{\alpha,\epsilon}(E)$ implies that $\epsilon^{\beta-\alpha}\mathcal{H}_{\alpha,\epsilon}(E) \to 0$ as $\epsilon \to 0$, so that $\mathcal{H}_\beta(E) = 0$.

(ii) This is the contrapositive of (i).

For the inequalities relating $\mathcal{H}_N$ and μ_e, we assume that both $\mathcal{H}_N(E)$ and $\mu_e(E)$ are finite.

For a fixed $\epsilon > 0$, there is a sequential covering $\{Q_n\}$ of E by dyadic cubes for which

$$\mu_e(E) \ge \sum_{n=1}^\infty \left(\frac{\mathrm{diam}(Q_n)}{\sqrt{N}}\right)^N - \epsilon = \frac{1}{N^{N/2}} \sum_{n=1}^\infty \big(\mathrm{diam}(Q_n)\big)^N - \epsilon.$$

Without loss of generality, we assume that $\operatorname{diam}(Q_n) < \epsilon$ for all N, by possibly subdividing any dyadic cube into a finite collection of dyadic cubes with sufficiently small diameters, so that $\{Q_n\} \subset \mathcal{E}_\epsilon$.

Then

$$\mu_e(E) \geq \frac{1}{N^{N/2}} \sum_{n=1}^{\infty} \big(\operatorname{diam}(Q_n)\big)^N - \epsilon \geq \frac{\mathcal{H}_{N,\epsilon}(E)}{N^{N/2}} - \epsilon.$$

We obtain by taking the limit as $\epsilon \to 0$ that

$$\frac{\mathcal{H}_N(E)}{N^{N/2}} \leq \mu_e(E).$$

For a fixed $\epsilon > 0$, there is a sequential covering $\{E_n\}$ of E by sets in $\mathcal{E}_\epsilon$ for which

$$\mathcal{H}_{N,\epsilon}(E) \geq \sum_{n=1}^{\infty} \big(\operatorname{diam}(E_n)\big)^N - \frac{\epsilon}{2}.$$

Since E_n is contained in a closed ball B_n of $\operatorname{diam}(E_n)$, there are finitely many dyadic cubes $\{Q_{n,j}\}$, $j = 1, \ldots, j_n$ which cover E_n.

We choose such a finite covering of E_n so that

$$\big(\operatorname{diam}(E_n)\big)^N = \frac{2^N}{\kappa_N} \operatorname{Vol}(B_n) \geq \frac{2^N}{\kappa_N} \sum_{j=1}^{j_n} \left(\frac{\operatorname{diam}(Q_{n,j})}{\sqrt{N}}\right)^N - \frac{\epsilon}{2^{n+1}}.$$

[The equality is proven in the Appendix of this lecture note.]

Thus we obtain

$$\begin{aligned}
\mathcal{H}_{N,\epsilon}(E) &\geq \frac{2^N}{\kappa_N} \sum_{n=1}^{\infty} \sum_{j=1}^{j_n} \left(\frac{\operatorname{diam}(Q_{n,j})}{\sqrt{N}}\right)^N - \epsilon \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} - \frac{\epsilon}{2} \\
&= \frac{2^N}{\kappa_N} \sum_{n=1}^{\infty} \sum_{j=1}^{j_n} \left(\frac{\operatorname{diam}(Q_{n,j})}{\sqrt{N}}\right)^N - \epsilon \\
&\geq \frac{2^N \mu_e(E)}{\kappa_N} - \epsilon.
\end{aligned}$$

We obtain by taking the limit as $\epsilon \to 0$ that $\mu_e(E) \leq \kappa_n \mathcal{H}_N(E)/2^N$. $\square$

We call $\mathcal{H}_\alpha$ the **Hausdorff outer measure**.

Recall the definition of the distance $\operatorname{dist}(E, F)$ between two nonempty subsets E and F of $\mathbb{R}^N$:

$$\operatorname{dist}(E, F) = \inf\{d(x, y) : x \in E, y \in F\}.$$

Proposition 5.2. If $E, F \in \mathcal{P}(\mathbb{R}^N)$ satisfy $\operatorname{dist}(E, F) > 0$, then for all $\alpha > 0$ there holds $\mathcal{H}_\alpha(E \cup F) = \mathcal{H}_\alpha(E) + \mathcal{H}_\alpha(F)$.

Proof. Since $\mathcal{H}_\alpha(E \cup F) \leq \mathcal{H}_\alpha(E) + \mathcal{H}_\alpha(F)$ by subadditivity, it remains to show that the opposite inequality holds when $\operatorname{dist}(E, F) > 0$.

We assume that $\mathcal{H}_\alpha(E \cup F) < \infty$, for otherwise equality holds trivially.

For $\delta = \text{dist}(E, F)$, we choose $\epsilon < \delta/2$.

Then there is a sequential covering $\{G_n\}$ of $E \cup F$ by sets in $\mathcal{E}_\epsilon$ for which

$$\mathcal{H}_{\alpha,\epsilon}(E \cup F) \geq \sum_{n=1}^{\infty} \big(\text{diam}(G_n)\big)^\alpha - \epsilon.$$

Since $\text{dist}(E, F) = \delta > 2\epsilon$, each G_n containing some point of E satisfies $G_n \cap F = \emptyset$.

Similarly, each G_m containing some point of F satisfies $G_m \cap E = \emptyset$.

These implies that the sequential covering $\{G_n\}$ splits into a sequential covering $\{E_k\}$ of E and a sequential covering $\{F_l\}$ of F. [In this splitting, it may happen that all but finitely many of the E_k or the F_l (but not both) are the empty set.]

Thus

$$\begin{aligned}\mathcal{H}_{\alpha,\epsilon}(E \cup F) &\geq \sum_{k=1}^{\infty} \big(\text{diam}(E_k)\big)^\alpha + \sum_{l=1}^{\infty} \big(\text{diam}(F_l)\big)^\alpha - \epsilon \\ &\geq \mathcal{H}_{\alpha,\epsilon}(E) + \mathcal{H}_{\alpha,\epsilon}(F) - \epsilon.\end{aligned}$$

Taking the limit as $\epsilon \to 0$ gives the desired inequality. $\square$

Appendix. The volume of a ball B in $\mathbb{R}^N$ with diameter d is

$$\mathrm{Vol}(B) = \begin{cases} \dfrac{(2\pi)^{N/2} d^N}{2^N (2 \cdot 4 \cdots N)} & \text{if } N \text{ is even,} \\ \dfrac{2(2\pi)^{(N-1)/2} d^N}{2^N (1 \cdot 3 \cdots N)} & \text{if } N \text{ is odd.} \end{cases}$$

Thus the value of κ_N, the volume of the unit ball in $\mathbb{R}^N$ whose diameter is 2, is

$$\kappa_N = \begin{cases} \dfrac{(2\pi)^{N/2}}{2 \cdot 4 \cdots N} & \text{if } N \text{ is even,} \\ \dfrac{2(2\pi)^{(N-1)/2}}{1 \cdot 3 \cdots N} & \text{if } N \text{ is odd.} \end{cases}$$

Thus we obtain the formula for the volume of a ball B in $\mathbb{R}^N$ with diameter d:

$$\mathrm{Vol}(B) = \frac{\kappa_N d^N}{2^N}.$$