Math 541 Lecture #9
II.6: Constructing measures from outer measures

6. The Carathéodory Procedure. We shall now describe the Carathéodory procedure of constructing a complete measure from an outer measure.

Let μ_e be an outer measure on a set X.

For any $A, E \in \mathcal{P}(X)$ we have the set identity

$$A = (A \cap E) \cup (A - E)$$

for a union of disjoint sets, which by countable subadditivity of μ_e gives

$$\mu_e(A) \leq \mu_e(A \cap E) + \mu_e(A - E).$$

We say an element $E \in \mathcal{P}(X)$ is μ_e-**measurable** if for all $A \in \mathcal{P}(X)$ there holds

$$\mu_e(A) = \mu_e(A \cap E) + \mu_e(A - E).$$

We denote by $\mathcal{A}$ the collection of all μ_e-measurable elements of $\mathcal{P}(X)$.

Proposition 6.1. For an outer measure μ_e, the collection $\mathcal{A}$ has the following properties.

(i) $\emptyset \in \mathcal{A}$.
(ii) If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.
(iii) If $E \in \mathcal{P}(X)$ satisfies $\mu_e(E) = 0$, then $E \in \mathcal{A}$.
(iv) If $E_1, E_2 \in \mathcal{A}$, then $E_1 \cup E_2 \in \mathcal{A}$.
(v) If $E_1, E_2 \in \mathcal{A}$, then $E_1 - E_2 \in \mathcal{A}$.
(vi) If $E_1, E_2 \in \mathcal{A}$, then $E_1 \cap E_2 \in \mathcal{A}$.
(vii) If $\{E_n\}$ is a countable collection of pairwise disjoint sets in $\mathcal{A}$, then for all $A \in \mathcal{P}(X)$ there holds

$$\lim_{m\to\infty} \mu_e\left(A \cap \left(\bigcup_{n=1}^{m} E_n\right)\right) = \mu_e\left(A \cap \left(\bigcup_{n=1}^{\infty} E_n\right)\right) = \sum_{n=1}^{\infty} \mu_e\big(A \cap E_n\big).$$

(viii) A countable union of sets in $\mathcal{A}$ is in $\mathcal{A}$.

Proof. (i) Since $\mu_e(A \cap \emptyset) + \mu_e(A - \emptyset) = \mu_e(\emptyset) + \mu_e(A) = \mu_e(A)$ for all $A \in \mathcal{P}(X)$, we have that $\emptyset \in \mathcal{A}$.

(ii) For $E \in \mathcal{A}$ we have $\mu_e(A) = \mu_e(A \cap E) + \mu_e(A - E) = \mu_e(A \cap E) + \mu_e(A \cap E^c)$ for all $A \in \mathcal{P}(X)$.

Since $A \cap E = A - E^c$ we obtain $\mu_e(A) = \mu_e(A - E^c) + \mu_e(A \cap E^c)$ for all $A \in \mathcal{P}(X)$, which implies that $E^c \in \mathcal{A}$.

(iii) If $\mu_e(E) = 0$, then for all $A \in \mathcal{P}(X)$ we have $0 \leq \mu_e(A \cap E) \leq \mu_e(E)$ and $\mu(A - E) \leq \mu_e(A)$ by monotonicity, so that for all $A \in \mathcal{P}(X)$ there holds

$$\mu_e(A \cap E) + \mu_e(A - E) \leq \mu_e(E) + \mu_e(A) = 0 + \mu_e(A) = \mu_e(A).$$

Thus $\mu_e(A) = \mu_e(A \cap E) + \mu_e(A - E)$ and so $E \in \mathcal{A}$.

(iv) For $E_1, E_2 \in \mathcal{A}$ we have for any $A \in \mathcal{P}(X)$ that

$$\mu_e(A) \geq \mu_e(A \cap E_1) + \mu_e(A - E_1),$$
$$\mu_e(A - E_1) \geq \mu_e((A - E_1) \cap E_2) + \mu_e((A - E_1) - E_2),$$

where the second inequality holds because $A - E_1 \in \mathcal{P}(X)$.

Because of the common summand $\mu_e(A - E_1)$ in these inequalities we get

$$\mu_e(A) \geq \mu_e(A \cap E_1) + \mu_e((A - E_1) \cap E_2) + \mu_e((A - E_1) - E_2).$$

By the subadditivity of μ_e we obtain

$$\mu_e(A) \geq \mu_e\big((A \cap E_1) \cup \big((A - E_1) \cap E_2\big)\big) + \mu_e((A - E_1) - E_2).$$

By the set identities $(A \cap E_1) \cup ((A - E_1) \cap E_2) = A \cap (E_1 \cup E_2)$ and $(A - E_1) - E_2 = A - (E_1 \cup E_2)$, we have

$$\mu_e(A) \geq \mu_e(A \cap (E_1 \cup E_2)) + \mu_e(A - (E_1 \cup E_2).$$

Thus $E_1 \cup E_2 \in \mathcal{A}$.

(v) Using the set identity $E_1 - E_2 = E_1 \cap E_2^c = (E_1^c \cup E_2)^c$ and (ii) and (iv), we have that $E_1 - E_2 \in \mathcal{A}$ whenever $E_1, E_2 \in \mathcal{A}$.

(vi) Using the set identity $E_1 \cap E_2 = (E_1^c \cup E_2^c)^c$ and (ii) and (iv), we have that $E_1 \cap E_2 \in \mathcal{A}$ whenever $E_1, E_2 \in \mathcal{A}$.

(vii) For a countable collection $\{E_n\}$ of pairwise disjoint sets in $\mathcal{A}$, set $B_k = \cup_{j=1}^k E_j$.

By pairwise disjointness of $\{E_n\}$ we have that $B_{k+1} - B_k = E_{k+1}$ for all k.

Let $A \in \mathcal{P}(X)$.

For $k = 1$, we have $\mu_e(A \cap B_1) = \sum_{j=1}^1 \mu_e(A \cap E_j)$.

Suppose that for $k \geq 1$ we have $\mu_e(A \cap B_k) = \sum_{j=1}^k \mu_e(A \cap E_j)$.

By (iv) we have $B_k \in \mathcal{A}$, so that

$$\begin{aligned} \mu_e(A \cap B_{k+1}) &= \mu_e\big((A \cap B_{k+1}) \cap B_k\big) + \mu_e\big((A \cap B_{k+1}) - B_k\big) \\ &= \mu_e\big(A \cap B_k\big) + \mu_e(A \cap E_{k+1}) \end{aligned}$$

By induction there holds $\mu_e(A \cap B_k) = \sum_{j=1}^k \mu_e(A \cap E_j)$ for all $k \in \mathbb{N}$.

By subadditivity and monotonicity of μ_e, and the induction above, we have for all $m \in \mathbb{N}$ that

$$\begin{aligned} \sum_{n=1}^{\infty} \mu_e(A \cap E_n) &\geq \mu_e\left(\bigcup_{n=1}^{\infty}(A \cap E_n)\right) = \mu_e\left(A \cap \left(\bigcup_{n=1}^{\infty} E_n\right)\right) \\ &\geq \mu_e\left(A \cap \left(\bigcup_{n=1}^{m} E_n\right)\right) = \sum_{n=1}^{m} \mu_e(A \cap E_n). \end{aligned}$$

Letting $m \to \infty$ forces the inequalities to be equalities, giving the result.

(viii) We may assume that $\{E_n\}$ are pairwise disjoint by replacing $\{E_n\}$ by $\{D_n\}$ where $D_1 = E_1$ and $D_n = E_{n+1} - \cup_{j=1}^n E_n$ if needed since each $D_n \in \mathcal{A}$ and $\cup D_n = \cup E_n$.

Each finite union $\cup_{n=1}^m E_n$ belongs to $\mathcal{A}$ so that for all $A \in \mathcal{P}(X)$ there holds

$$\mu_e(A) = \mu_e\left(A \cap \left(\bigcup_{n=1}^m E_n\right)\right) + \mu_e\left(A - \bigcup_{n=1}^m E_n\right).$$

Since $A - \cup_{n=1}^m E_n \supset A - \cup_{n=1}^\infty E_n$, we have by monotonicity that

$$\mu_e\left(A - \bigcup_{n=1}^m E_n\right) \geq \mu_e\left(A - \bigcup_{n=1}^\infty E_n\right).$$

By (vii) we have that

$$\lim_{m\to\infty} \mu_e\left(A \cap \left(\bigcup_{n=1}^m E_n\right)\right) = \mu_e\left(A \cap \left(\bigcup_{n=1}^\infty E_n\right)\right).$$

Thus

$$\mu_e(A) \geq \mu_e\left(A \cap \left(\bigcup_{n=1}^\infty E_n\right)\right) + \mu_e\left(A - \bigcup_{n=1}^\infty E_n\right).$$

This implies that $\cup E_n$ belongs to $\mathcal{A}$. □

Proposition 6.2 (Carathéodory). The restriction of μ_e to $\mathcal{A}$ is a complete measure.

Proof. The nonempty set $\mathcal{A}$ is a algebra by parts (i), (ii), and (iv) of Proposition 6.1.

The algebra $\mathcal{A}$ is a σ-algebra by part (viii) of Proposition 6.1.

The outer measure μ_e restricted to $\mathcal{A}$ is a measure, where countable additivity follows from part (vii) of Proposition 6.1 with $A = \cup E_n$ for $\{E_n\}$ in $\mathcal{A}$ being pairwise disjoint:

$$\begin{aligned}
\mu_e\left(\bigcup_{n=1}^\infty E_n\right) &= \mu_e\left(\left(\bigcup_{m=1}^\infty E_m\right) \cap \left(\bigcup_{n=1}^\infty E_n\right)\right) \\
&= \sum_{n=1}^\infty \mu_e\left(\left(\bigcup_{m=1}^\infty E_m\right) \cap E_n\right) \\
&= \sum_{n=1}^\infty \mu_e(E_n).
\end{aligned}$$

The completeness of μ_e restricted to $\mathcal{A}$ is by part (iii) of Proposition 6.1. □