Math 541 Lecture #10
II.7: The Lebesgue-Stieltjes measure on $\mathbb{R}$
II.8: The Hausdorff measure on $\mathbb{R}^N$

7. The Lebesgue-Steltjes measure on $\mathbb{R}$. Recall that the Lebesgue-Stieltjes outer measure $\mu_{f,e}$ is generated by a right-continuous monotone increasing function $f:\mathbb{R}\to\mathbb{R}$ and the sequential covering $\mathcal{Q}$ of $\mathbb{R}$ consisting of open intervals:

$$\mu_{f,e}(E) = \inf\left\{\sum_{n=1}^\infty \lambda_f(Q_n) : \{Q_n\}\in\mathcal{Q}, E\subset\bigcup_{n=1}^\infty Q_n\right\}$$

where $\lambda_f\big((a,b)\big) = f(b)-f(a)$.

By the Carathéodory procedure, the outer measure $\mu_{f,e}$ determines a σ-algebra $\mathcal{A}_f$ and a complete measure μ_f on $\mathcal{A}_f$, known as the Lebesgue-Stieltjes measure on $\mathbb{R}$.

Recall that the Borel σ-algebra $\mathcal{B}_\mathbb{R}$ is the smallest σ-algebra containing all of the open sets of $\mathbb{R}$.

We will show that $\mathcal{A}_f$ contains $\mathcal{B}_\mathbb{R}$.

Lemma 7.2. For any open interval I and any interval of the type $(\alpha,\beta]$ there holds

$$\lambda_f(I)\geq \mu_{f,e}\big(I\cap(\alpha,\beta]\big) + \mu_{f,e}(I-(\alpha,\beta]).$$

Proof. For $I=(a,b)$ we consider the case $a\leq\alpha<\beta<b$, i.e., $(\alpha,\beta]\subset(a,b)$.

We have $I\cap(\alpha,\beta]) = (\alpha,\beta]$, and $I-(\alpha,\beta] = (a,\alpha]\cup(\beta,b)$.

Let ϵ and η be positive real numbers.

Then

$$\begin{aligned}(\beta,b)&\subset\lim_{\epsilon\to0}(\beta+\epsilon,b),\\ (\alpha,\beta]&\subset\lim_{\epsilon\to0}\lim_{\eta\to0}(\alpha+\eta,\beta+\epsilon),\\ (a,\alpha]&\subset\lim_{\eta\to0}(a,\alpha+\eta],\end{aligned}$$

By the right-continuity of f and the definition and properties of the Lebesgue-Stieltjes outer measure we have

$$\begin{aligned}\lambda_f(I) &= f(b)-f(a)\\ &= \lim_{\epsilon\to0}\big(f(b)-f(\beta+\epsilon)\big) + \lim_{\epsilon\to0}\lim_{\eta\to0}\big(f(\beta+\epsilon)-f(\alpha+\eta)\big)\\ &\quad+\lim_{\eta\to0}\big(f(\alpha+\eta)-f(a)\big)\\ &\geq \mu_{f,e}\big((\beta,b)\big)+\mu_{f,e}\big((\alpha,\beta]\big)+\mu_{f,e}\big((a,\alpha]\big)\\ &\geq \mu_{f,e}\big(I\cap(\alpha,\beta]\big)+\mu_{f,e}\big(I-(\alpha,\beta]\big).\end{aligned}$$

The other two cases, $\alpha\leq a<\beta<b$ and $a\leq\alpha<b<\beta$, are handled similarly. $\square$

Proposition 7.1. The σ-algebra $\mathcal{A}_f$ contains $\mathcal{B}_{\mathbb{R}}$.

Proof. It suffices to show that intervals of the form $(\alpha, \beta]$ belong to $\mathcal{A}_f$, since any open interval is a countable union of "half-open" intervals:

$$(a,b) = \bigcup_{n=1}^{\infty} (a, b - 1/n].$$

An interval of the form $(\alpha, \beta]$ belongs to $\mathcal{A}_f$ when for all $A \in \mathcal{P}(\mathbb{R})$ it satisfies

$$\mu_{f,e}(A) \geq \mu_{f,e}\big(A \cap (\alpha, \beta]\big) + \mu_{f,e}\big(A - (\alpha, \beta]\big).$$

WLOG we assume that $\mu_{f,e}(A) < \infty$ (for the inequality is true when it is infinite).

For a fixed $\epsilon > 0$ choose a sequential covering of A by open intervals $\{I_n\}$ so that

$$\mu_{f,e}(A) + \epsilon \geq \sum_{n=1}^{\infty} \lambda_f(I_n).$$

By Lemma 7.2 we know for each $n \in \mathbb{N}$ that

$$\lambda_f(I_n) \geq \mu_{f,e}\big(I_n \cap (\alpha, \beta]\big) + \mu_{f,e}\big(I_n - (\alpha, \beta]\big).$$

Hence

$$\begin{aligned}
\mu_{f,e}(A) + \epsilon &\geq \sum_{n=1}^{\infty} \mu_{f,e}\big(I_n \cap (\alpha, \beta]\big) + \sum_{n=1}^{\infty} \mu_{f,e}\big(I_n - (\alpha, \beta]\big) \\
&\geq \mu_{f,e}\left(\bigcup_{n=1}^{\infty} \big(I_n \cap (\alpha, \beta]\big)\right) + \mu_{f,e}\left(\bigcup_{n=1}^{\infty} \big(I_n - (\alpha, \beta]\big)\right) \\
&\geq \mu_{f,e}\left(A \cap (\alpha, \beta]\right) + \mu_{f,e}\left(A - (\alpha, \beta]\right),
\end{aligned}$$

where countable subadditivity and monotonicity of $\mu_{f,e}$ have been used.

Since $\epsilon > 0$ is arbitrary, we obtain that $(\alpha, \beta] \in \mathcal{A}_f$. $\square$

7.1: Borel Measures. A measure μ in $\mathbb{R}^N$ is a **Borel measure** if the σ-algebra of the domain of μ contains the Borel sets $\mathcal{B}_{\mathbb{R}^N}$ in $\mathbb{R}^N$.

Proposition 7.1 states that the Lebesgue-Stieltjes measure μ_f is a Borel measure in $\mathbb{R}$.

8: The Hausdorff Measure on $\mathbb{R}^N$. Recall for each fixed $\alpha > 0$, that the Hausdorff outer measure $\mathcal{H}_\alpha$ on $\mathbb{R}^n$ generates a σ-algebra $\mathcal{A}_\alpha$ and a complete measure μ_α in $\mathbb{R}^n$.

Theorem. For all $\alpha > 0$, the Hausdorff measure μ_α is a Borel measure.

Proof. It suffices to show that every closed set $E \subset \mathbb{R}^N$ belongs to $\mathcal{A}_\alpha$, because each open set is a countable union of closed sets (an F_σ set).

So for each closed E we will show for all $A \in \mathcal{P}(\mathbb{R}^N)$ that

$$\mathcal{H}_\alpha(A) \geq \mathcal{H}_\alpha(A \cap E) + \mathcal{H}_\alpha(A - E).$$

We assume that $\mathcal{H}_\alpha(A) < \infty$, and for each $n \in \mathbb{N}$ set

$$E_n = \left\{ x \in \mathbb{R}^N : \operatorname{dist}\{x, E\} \leq \frac{1}{n} \right\}.$$

We have $E \subset E_n$ for all n.

For each n we have $A = (A \cap E_n) \cup (A - E_n)$, so that

$$\mathcal{H}_\alpha(A) = \mathcal{H}_\alpha\big((A \cap E_n) \cup (A - E_n)\big) \geq \mathcal{H}_\alpha\big((A \cap E) \cup (A - E_n)\big),$$

where we have used monotonicity of $\mathcal{H}_\alpha$ and $A \cap E \subset A \cap E_n$.

Since $\operatorname{dist}\{A \cap E; A - E_n\} \geq 1/n$ for each n, we have by Proposition 5.2 for each n that

$$\mathcal{H}_\alpha\big((A \cap E) \cup (A - E_n)\big) = \mathcal{H}_\alpha(A \cap E) + \mathcal{H}_\alpha(A - E_n).$$

Thus we have

$$\mathcal{H}_\alpha(A) \geq \mathcal{H}_\alpha(A \cap E) + \mathcal{H}_\alpha(A - E_n).$$

Using the set identity $A - E = (A - E_n) \cup (A \cap (E_n - E))$ and the subadditivity of $\mathcal{H}_\alpha$ we get

$$\mathcal{H}_\alpha(A - E) \leq \mathcal{H}_\alpha(A - E_n) + \mathcal{H}_\alpha\big(A \cap (E_n - E)\big).$$

Hence

$$\mathcal{H}_\alpha(A - E_n) \geq \mathcal{H}_\alpha(A - E) - \mathcal{H}_\alpha\big(A \cap (E_n - E)\big).$$

Thus for all n we have

$$\mathcal{H}_\alpha(A) \geq \mathcal{H}_\alpha(A \cap E) + \mathcal{H}_\alpha(A - E) - \mathcal{H}_\alpha\big(A \cap (E_n - E)\big).$$

The desired result follows if $\lim_{n\to\infty} \mathcal{H}_\alpha\big(A \cap (E_n - E)\big) = 0$. (This is Lemma 8.2 in the Ed.1 of text; Lemma 8.1 in Ed. 2.)

For $j \geq n$, the sets

$$F_j = \left\{x \in A : \frac{1}{j+1} < \operatorname{dist}\{x; E\} \leq \frac{1}{j}\right\}$$

satisfy

$$A \cap (E_n - E) = \bigcup_{j=n}^{\infty} F_j.$$

By subadditivity we have

$$\mathcal{H}_\alpha\big(A \cap (E_n - E)\big) \leq \sum_{j=n}^{\infty} \mathcal{H}_\alpha(F_j).$$

The conclusion $\lim_{n\to\infty} \mathcal{H}_\alpha\big(A \cap (E_n - E)\big) = 0$ follows if the series $\sum_{j=1}^{\infty} \mathcal{H}_\alpha(F_j)$ of nonnegative terms converges.

For $i, j \in \mathbb{N}$ both even or both odd, we have $\operatorname{dist}\{F_i; F_j\} > 0$.

Thus applying Proposition 5.2 we have for any $m \in \mathbb{N}$ that

$$\sum_{j=1}^{m} \mathcal{H}_\alpha(F_j) \leq \sum_{h=1}^{m} \mathcal{H}_\alpha(F_{2h}) + \sum_{h=0}^{m} \mathcal{H}_\alpha(F_{2h+1}) = \mathcal{H}_\alpha\left(\bigcup_{h=1}^{m} F_{2h}\right) + \mathcal{H}_\alpha\left(\bigcup_{h=0}^{m-1} F_{2h+1}\right).$$

The two unions are subsets of A, so by monotonicity we have $\sum_{j=1}^{m} \mathcal{H}_\alpha(F_j)$ is bounded above by $2\mathcal{H}_\alpha(A) < \infty$.

Thus the series $\sum_{j=1}^{\infty} \mathcal{H}_\alpha(F_j)$ converges. $\square$

Homework Problem 10A. Where did we used that E is closed in the above proof?