Math 541 Lecture #11
II.9: Extending measures from semialgebras to σ-algebras

9. Extending measures from semialgebras to σ-algebras. Recall the Carathéodory procedure generates from an outer measure μ_e (as determined by a sequential covering $\mathcal{Q}$ of X and a nonnegative function $\lambda : \mathcal{Q} \to \mathbb{R}^*$) a σ-algebra $\mathcal{A}$ and a (complete) measure μ on $\mathcal{A}$.

It may happen that $\mathcal{Q} \not\subset \mathcal{A}$ or that for some $E \in \mathcal{Q} \cap \mathcal{A}$ we have $\mu_e(E) \neq \lambda(E)$ (we saw the latter for Lebsegue-Stieltjes outer measure and Hausdorff outer measure).

We describe sufficient conditions on $\mathcal{Q}$ and λ by which we obtain $\mathcal{Q} \subset \mathcal{A}$ and $\mu_e(Q) = \lambda(Q)$ for all $Q \in \mathcal{Q}$.

Definition. A collection $\mathcal{Q}$ of subsets of X is a **semi-algebra** if

(i) the intersection of any two elements of $\mathcal{Q}$ is in $\mathcal{Q}$, and

(ii) the difference $Q_1 - Q_2$ of any two elements of $\mathcal{Q}$ is a finite pairwise disjoint union of elements of $\mathcal{Q}$.

Examples. (a) The collection of $\frac{1}{2}$-closed dyadic cubes in $\mathbb{R}^n$ is a semialgebra.

(b) The collection of open subintervals of $\mathbb{R}$ is not a semialgebra because property (ii) fails: the difference $(0, 3) - (1, 2) = (0, 1] \cup [2, 3)$ is not the finite pairwise disjoint union of open intervals.

Definition. Let $\mathcal{Q} \subset \mathcal{P}(X)$. A function $\lambda : \mathcal{Q} \to \mathbb{R}^*$ is **finitely additive** on $\mathcal{Q}$ if for any finite collection $Q_1, \ldots, Q_m$ of pairwise disjoint elements of $\mathcal{Q}$ whose union is in $\mathcal{Q}$, there holds

$$\lambda\left(\bigcup_{j=1}^m Q_j\right) = \sum_{j=1}^m \lambda(Q_j).$$

[We need to assume that the union is in $\mathcal{Q}$ so that λ is defined on it, because neither $\mathcal{Q}$ being a sequential covering nor $\mathcal{Q}$ a semialgebra guarantees that a union of elements of $\mathcal{Q}$ is in $\mathcal{Q}$.]

Examples. (a) The Euclidean volume of $\frac{1}{2}$-closed dyadic cubes is finitely additive.

(b) The Lebesgue-Stieltjes function λ_f is finitely additive.

(c) For the Hausdorff outer measure, the function $\lambda(E) = (\mathrm{diam}(E))^\alpha$ on $\mathcal{E}_\epsilon$ is not finitely additive.

Homework Problem 11A: Prove parts (b) and (c)

Definition. A function $\lambda : \mathcal{Q} \to \mathbb{R}^*$ is **countably subadditive** on $\mathcal{Q}$ if for any countable collection $\{Q_j\}$ of elements of $\mathcal{Q}$ whose union is in $\mathcal{Q}$, there holds

$$\lambda\left(\bigcup_{j=1}^\infty Q_j\right) \leq \sum_{j=1}^\infty \lambda(Q_j).$$

Propositon 9.1. Let $\mathcal{Q}$ be a sequential covering. If $\mathcal{Q}$ is a semialgebra and $\lambda : \mathcal{Q} \to \mathbb{R}^*$ is finitely additive on $\mathcal{Q}$, then $\mathcal{Q} \subset \mathcal{A}$. If, in addition, λ is countably subadditive on $\mathcal{Q}$, then $\mu_e(Q) = \lambda(Q)$ for all $Q \in \mathcal{Q}$.

Proof. Fix an arbitrary $Q \in \mathcal{Q}$, and select $A \subset X$.

If $\mu_e(A) = \infty$, then we have

$$\mu_e(A) \geq \mu_e(A \cap Q) + \mu_e(A - Q).$$

So suppose that $\mu_e(A) < \infty$.

For $\epsilon > 0$ there is a countable collection $\{Q_{\epsilon,n}\}$ in $\mathcal{Q}$ such that

$$\epsilon + \mu_e(A) \geq \sum_{n=1}^{\infty} \lambda(Q_{\epsilon,n}) \text{ and } A \subset \bigcup_{n=1}^{\infty} Q_{\epsilon,n}.$$

For each fixed n we have

$$Q_{\epsilon,n} = \big(Q_{\epsilon,n} \cap Q\big) \cup \big(Q_{\epsilon,n} - Q\big).$$

Since $\mathcal{Q}$ is a semialgebra, the intersection $Q_{\epsilon,n} \cap Q$ is in $\mathcal{Q}$; and the difference

$$Q_{\epsilon,n} - Q = \bigcup_{j=1}^{m_n} Q_{j,n}$$

where $Q_{j,n}$, $j = 1, \ldots, m_n$, are disjoint elements of $\mathcal{Q}$.

Hence each $Q_{\epsilon,n}$ is the disjoint union of finitely many elements of $\mathcal{Q}$.

By the assumed finite additivity of λ we have

$$\lambda(Q_{\epsilon,n}) = \lambda(Q_{\epsilon,n} \cap Q) + \sum_{j=1}^{m_n} \lambda(Q_{j,n}).$$

The elements of the collection $\{Q_{\epsilon,n} \cap Q\}$ are in $\mathcal{Q}$ and their union covers $A \cap Q$.

Also, the elements of the collection $\{Q_{j,n}\}$, for $j = 1, \ldots, m_n$ and $n \in \mathbb{N}$, are in $\mathcal{Q}$, and their union covers $A - Q$.

Thus by the definition of outer measure, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda(Q_{\epsilon,n}) &= \sum_{n=1}^{\infty} \lambda(Q_{\epsilon,n} \cap Q) + \sum_{n=1}^{\infty} \sum_{j=1}^{m_n} \lambda(Q_{j,n}) \\ &\geq \mu_e(A \cap Q) + \mu_e(A - Q).\end{aligned}$$

Therefore for all $\epsilon > 0$ we have

$$\epsilon + \mu_e(A) \geq \mu_e(A \cap Q) + \mu_e(A - Q).$$

This implies that $Q \in \mathcal{A}$.

To show that $\mu_e(Q) = \lambda(Q)$ for all $Q \in \mathcal{Q}$, we assume WLOG that $\mu_e(Q) < \infty$.

By the definition of the outer measure, we know that $\mu_e(Q) \leq \lambda(Q)$, so it suffices to show the opposite inequality.

Fix $\epsilon > 0$ and let $\{Q_{\epsilon,n}\}$ be a countable collection of elements in $\mathcal{Q}$ such that

$$\epsilon + \mu_e(Q) \geq \sum_{n=1}^{\infty} \lambda(Q_{\epsilon,n}) \text{ and } Q \subset \bigcup_{n=1}^{\infty} Q_{\epsilon,n}.$$

As done before, we have for each $n \in \mathbb{N}$ the existence of $Q_{j,n} \in \mathcal{Q}$, $j = 1, \ldots, m_n$, for which

$$\lambda(Q_{\epsilon,n}) = \lambda(Q_{\epsilon,n} \cap Q) + \sum_{j=1}^{m_n} \lambda(Q_{j,n}),$$

where the elements of $\{Q_{\epsilon,n} \cap Q\}$ belong to $\mathcal{Q}$; hence for each $n \in \mathbb{N}$,

$$\lambda(Q_{\epsilon,n}) \geq \lambda(Q_{\epsilon,n} \cap Q).$$

Since $Q \subset \cup_{n=1}^{\infty} Q_{\epsilon,n}$, we have that

$$Q = \bigcup_{n=1}^{\infty} \left(Q \cap Q_{\epsilon,n}\right),$$

which union belongs to $\mathcal{Q}$ because Q does.

By the assumed countable subadditivity of λ we have

$$\lambda(Q) = \lambda\left(\bigcup_{n=1}^{\infty}(Q \cap Q_{\epsilon,n})\right) \leq \sum_{n=1}^{\infty} \lambda(Q_{\epsilon,n} \cap Q) \leq \sum_{n=1}^{\infty} \lambda(Q_{\epsilon,n}) \leq \mu_e(Q) + \epsilon.$$

As this holds for any $\epsilon > 0$ we obtain the opposite inequality. □

Definition. A function λ is said to be a **measure on a semialgebra** $\mathcal{Q}$ if it satisfies

(i) the domain of λ is $\mathcal{Q}$,

(ii) λ is nonnegative real-valued on $\mathcal{Q}$,

(iii) λ is countably additive on $\mathcal{Q}$, in the sense that for a countable collection of pairwise disjoint $\{Q_i\}$ in $\mathcal{Q}$ whose union is on $\mathcal{Q}$ we have

$$\lambda\left(\bigcup_{i=1}^{\infty} Q_i\right) = \sum_{i=1}^{\infty} \lambda(Q_i),$$

(iv) $\lambda(Q) < \infty$ for some $Q \in \mathcal{Q}$.

Proposition A. If λ is a measure on a semialgebra $\mathcal{Q}$, then λ is finitely additive and countably subaddtive on $\mathcal{Q}$.

Homework problem 11B. Prove that a measure λ on a semialgebra is finitely additive. [A proof that λ is countably subadditive is found in the appendix of Lecture Note #12.]

Combining Proposition 9.1 and Proposition A gives: for a sequential covering $\mathcal{Q}$ of X and a function λ defined on $\mathcal{Q}$, if $\mathcal{Q}$ is a semialgebra and λ is a measure on $\mathcal{Q}$, then $\mathcal{Q} \subset \mathcal{A}$ and $\mu_e(Q) = \lambda(Q)$ for all $Q \in \mathcal{Q}$, and the (complete) measure μ on the σ-algebra $\mathcal{A}$, which is the restriction of the outer measure μ_e to $\mathcal{A}$, is an extension of the measure λ on $\mathcal{Q}$.