Math 541 Lecture #12
II.10: Necessary and Sufficient Conditions for Measurability Part I

10. Necessary and sufficient conditions for measurability. Recall that a measure λ on a semilagebra $\mathcal{Q}$ (that is also a sequential covering of X) generates a σ-algebra $\mathcal{A}$ and a measure μ on $\mathcal{A}$ (the restriction of the outer measure μ_e to $\mathcal{A}$).

That is, the measure μ on $\mathcal{A}$ is an extension of the measure λ on $\mathcal{Q}$ in that $\mathcal{Q} \subset \mathcal{A}$ and $\lambda(E) = \mu(E)$ for all $E \in \mathcal{Q}$.

Recall that $\mathcal{A}$ consists of those sets E in X for which

$$\mu_e(A) \geq \mu_e(A \cap E) + \mu_e(A - E)$$

for all sets A in X; we say the elements of $\mathcal{A}$ are μ-measurable (what we called μ_e-measurable sets before).

We let $\{X, \mathcal{A}, \mu\}$ be the measure space generated by the measure λ on the sequential covering and semialgebra $\mathcal{Q}$.

We will describe sufficient and necessary conditions for the μ-measurable sets in terms of sets derived from elements of $\mathcal{Q}$.

Denote by $\mathcal{Q}_\sigma$ the collection of all sets that are countable unions of elements of $\mathcal{Q}$.

Note that $\mathcal{Q}_\sigma \subset \mathcal{A}$ since $\mathcal{Q} \subset \mathcal{A}$ and the latter is a σ-algebra.

Lemma. For each element $E = \cup Q_n$ of $\mathcal{Q}_\sigma$ there exists a countable collection of pairwise disjoint set $\{D_n\}$ in $\mathcal{A}$ such that $D_n \subset Q_n$ for all $n \in \mathbb{N}$, and $E = \cup D_n$, and each D_n is a finite union of pairwise disjoint elements of $\mathcal{Q}$.

Proof. Recall that we have seen before that any countable union can be rewritten as a countable union of disjoint sets: $E = \cup D_n$ where $D_1 = Q_1$, and

$$D_n = Q_n - \bigcup_{j=1}^{n-1} Q_j = \bigcap_{j=1}^{n-1} (Q_n - Q_j),\ n \in \mathbb{N}.$$

Here $D_n \subset Q_n$, and since $\mathcal{Q} \subset \mathcal{A}$ and the latter is a σ-algebra, we have $D_n \in \mathcal{A}$.

Because $\mathcal{Q}$ is a semialgebra, each difference $Q_n - Q_j$ is the finite disjoint union of elements of $\mathcal{Q}$.

Then for each n and j there exist $k(j)$ disjoint sets $P_{j,1}, \dots, P_{j,k(j)}$ in $\mathcal{Q}$ (with k depending on n as well but suppressed in the notation for the sake of clarity) such that

$$Q_n - Q_j = \bigcup_{l=1}^{k(j)} P_{j,l}.$$

We then have that

$$D_n = \bigcap_{j=1}^{n-1} \left(\bigcup_{l=1}^{k(j)} P_{j,l} \right).$$

Using the distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, we have

$$(A \cup B) \cap (C \cup D) = (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D).$$

Applying this repeatedly gives

$$\begin{aligned} D_n &= (P_{1,1} \cup P_{1,2} \cup \cdots \cup P_{1,k(1)}) \cap (P_{2,1} \cup P_{2,2} \cup \cdots \cup P_{2,k(2)}) \\ &\quad \cap \cdots \cap (P_{n-1,1} \cup P_{n-1,2} \cup \cdots \cup P_{n-1,k(n-1)}) \\ &= (P_{1,1} \cap P_{2,1} \cap \cdots \cap P_{n-1,1}) \cup (P_{1,1} \cap P_{2,1} \cap \cdots \cap P_{n-1,2}) \\ &\quad \cup \cdots \cup (P_{1,k(1)} \cap P_{2,k(2)} \cap \cdots \cap P_{n-1,k(n-1)}). \end{aligned}$$

That is, we have a union of $k(1)k(2)\cdots k(n-1)$ sets of the form

$$\bigcap_{j=1}^{n-1} P_{j,s(j)},$$

where for each j the value of $s(j)$ is chosen from $\{1, 2, \ldots, k(j)\}$.

Because a semialgebra is also "closed" under intersections, we have that each of these sets $\cap_{j=1}^{n-1} P_{j,s(j)}$ belongs to $\mathcal{Q}$.

For another set $\cap_{j=1}^{n-1} P_{j,r(j)}$, where $r(j) \in \{1, 2, \ldots, k(j)\}$ differs from $s(j)$ for some $j = \hat{j}$, we have that $\cap_{j=1}^{n-1} P_{j,s(j)}$ and $\cap_{j=1}^{n-1} P_{j,r(j)}$ are disjoint because the first is a subset of $P_{\hat{j},s(\hat{j})}$, the second a subset of $P_{\hat{j},r(\hat{j})}$, while $P_{\hat{j},s(\hat{j})}$ and $P_{\hat{j},r(\hat{j})}$ are disjoint.

Thus each D_n is a disjoint union of elements of $\mathcal{Q}$. □

Denote by $\mathcal{Q}_{\sigma\delta}$ the collection of sets that are countable intersections of elements of $\mathcal{Q}_\sigma$.

Note that $\mathcal{Q}_{\sigma\delta} \subset \mathcal{A}$ because $\mathcal{Q}_\sigma \subset \mathcal{A}$ and the latter is a σ-algebra.

Proposition 10.1. If $E \subset X$ is of finite outer measure, then for each $\epsilon > 0$ there exists $E_{\sigma,\epsilon} \in \mathcal{Q}_\sigma$ such that

$$E \subset E_{\sigma,\epsilon} \text{ and } \mu_e(E) \geq \mu(E_{\sigma,\epsilon}) - \epsilon.$$

Moreover, there exists a set $E_{\sigma\delta} \in \mathcal{Q}_{\sigma\delta}$ such that

$$E \subset E_{\sigma\delta} \text{ and } \mu_e(E) = \mu(E_{\sigma\delta}).$$

Proof. For a given $\epsilon > 0$ there exists $\{Q_{n,\epsilon}\}$ in $\mathcal{Q}$ such that

$$\mu_e(E) + \epsilon \geq \sum_{n=1}^{\infty} \lambda(Q_{n,\epsilon}), \quad E \subset \bigcup Q_{n,\epsilon}.$$

Set $E_{\sigma,\epsilon} = \cup Q_{n,\epsilon}$.

By the Lemma we can replace $\cup Q_{n,\epsilon}$ by a union of disjoint elements $D_{n,\epsilon} \in \mathcal{A}$ satisfying $D_{n,\epsilon} \subset Q_{n,\epsilon}$ (and where each $D_{n,\epsilon}$ is a finite disjoint union of elements of $\mathcal{Q}$ – something we will not need here).

Then $\mu(D_{n,\epsilon}) = \mu_e(D_{n,\epsilon}) \leq \mu_e(Q_{n,\epsilon}) \leq \lambda(Q_{n,\epsilon})$ because $D_{n,\epsilon} \in \mathcal{A}$ and $D_{n,\epsilon} \subset Q_{n,\epsilon} \in \mathcal{Q}$.

Thus by the pairwise disjointness of $\{D_{n,\epsilon}\}$ and the countable additivity of μ we get

$$\sum_{n=1}^{\infty} \lambda(Q_{n,\epsilon}) \geq \sum_{n=1}^{\infty} \mu(D_{n,\epsilon}) = \mu\left(\bigcup_{n=1}^{\infty} D_{n,\epsilon}\right) = \mu(E_{\sigma,\epsilon}).$$

This gives

$$\mu_e(E) + \epsilon \geq \mu(E_{\sigma,\epsilon}).$$

Now for each $n \in \mathbb{N}$, there is $E_{\sigma,1/n} \in \mathcal{Q}_\sigma$ that satifies

$$\mu(E_{\sigma,1/n}) - \frac{1}{n} \leq \mu_e(E) \leq \mu_e(E_{\sigma,1/n}) = \mu(E_{\sigma,1/n}),$$

the last inequality holds by $E \subset E_{\sigma,1/n}$ and the monotonicity of μ_e, while the last equality holds by $\mu = \mu_e$ on $\mathcal{A}$.

The set $E_{\sigma\delta} = \cap E_{\sigma,1/n} \in \mathcal{Q}_{\sigma\delta}$ contains E because each $E_{\sigma,1/n}$ does.

Homework Problem 12A. Prove that $E_{\sigma\delta}$ satisfies $\mu_e(E) = \mu(E_{\sigma\delta})$. $\square$

Appendix. Proof that a measure λ on semialgebra is countably subadditive.

Suppose for $\{Q_n\} \subset \mathcal{Q}$ that

$$Q = \bigcup_{n=1}^{\infty} Q_n \in \mathcal{Q}.$$

By Lemma in this Lecture Note there exist disjoint subsets $D_n \subset Q_n$ such that

$$Q = \bigcup_{n=1}^{\infty} D_n,$$

where $D_1 = Q_1$ and for each $n \geq 2$, there exists $k(n) \in \mathbb{N}$ and sets $P_{j,l} \in \mathcal{Q}$ for all $j = 1, \ldots, n$ and $l = 1, \ldots, k(n)$ such that

$$D_n = Q_n - \bigcup_{j=1}^{n-1} Q_j = \bigcap_{j=1}^{n-1} (Q_n - Q_j) = \bigcap_{j=1}^{n-1} \left(\bigcup_{l=1}^{k(j)} P_{j,l} \right) = \bigcup_{s} \left(\bigcap_{j=1}^{n-1} P_{j,s(j)} \right).$$

Set

$$E_{n,s} = \bigcap_{j=1}^{n-1} P_{j,s(j)} \in \mathcal{Q}.$$

For different functions s, the sets $E_{n,s}$ are pairwise disjoint. Then

$$Q = \bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} \bigcup_{s} E_{n,s}$$

is a countable union of pairwise disjoint sets. Hence by countable additivity we have

$$\lambda(Q) = \sum_{n=1}^{\infty} \sum_{s} \lambda(E_{n,s}).$$

If we can show the claim that for all n there holds

$$\sum_{s} \lambda(E_{n,s}) \leq \lambda(Q_n)$$

then we obtain the desired countable subadditivity,

$$\lambda\left(\bigcup_{n=1}^{\infty} Q_n \right) \leq \sum_{n=1}^{\infty} \lambda(Q_n).$$

It remains to establish the claim. To this end we consider $Q_n - D_n$, i.e.,

$$\begin{aligned} Q_n - \bigcup_{s} E_{n,s} &= Q_n \cap \left(\bigcup_{s} E_{n,s} \right)^c = Q_n \cap \left(\bigcap_{s} E_{n,s}^c \right) \\ &= \bigcap_{s} (Q_n \cap E_{n,s}^c) = \bigcap_{s} (Q_n - E_{n,s}). \end{aligned}$$

Since $Q_n \in \mathcal{Q}$ and $E_{n,s} \in \mathcal{Q}$, there exists $m(n,s) \in \mathbb{N}$ and disjoint $A_{n,s,t} \in \mathcal{Q}$ for $t = 1, \ldots, m(n,s)$ such that

$$Q_n - E_{n,s} = \bigcup_{t=1}^{m(n,s)} A_{n,s,t}.$$

Thus

$$Q_n - \bigcup_s E_{n,s} = \bigcap_s \left(\bigcup_{t=1}^{m(n,s)} A_{n,s,t} \right).$$

The finite intersection of the finite disjoint union can be written as a finite disjoint union of finite intersections of the sets $A_{n,s,t}$ where these finite intersections belong to $\mathcal{Q}$. (Recall the proof of the Lemma in this Lecture Note wherein the order of the intersection and union were reversed.) That is, there exist finitely many disjoint elements $B_1, \ldots, B_u$ of $\mathcal{Q}$ such that

$$Q_n - \bigcup_s E_{n,s} = \bigcup_{i=1}^{u} B_i.$$

Because $D_n \subset Q_n$, we obtain

$$Q_n = \bigcup_s E_{n,s} \cup \bigcup_{i=1}^{u} B_i.$$

Thus by finite additivity we have

$$\lambda(Q_n) = \sum_s \lambda(E_{n,s}) + \sum_{i=1}^{n} \lambda(B_i).$$

Since $\lambda \geq 0$, we obtain

$$\lambda(Q_n) \geq \sum_s \lambda(E_{n,s}),$$

which establishes the claim.