Math 541 Lecture #13
II.10: Necessary and Sufficient Conditions for Measurability Part II

10. Necessary and sufficient conditions for measurability. We continue with approximating μ-measurable sets.

Proposition 10.2. Let $\{X, \mathcal{A}, \mu\}$ be the measure space generated by the measure λ on the sequential covering and semialgebra $\mathcal{Q}$. A set $E \subset X$ of finite outer measure is μ-measurable if and only if for every $\epsilon > 0$ there exists a set $E_{\sigma,\epsilon}$ in $\mathcal{Q}_\sigma$ such that

$$E \subset E_{\sigma,\epsilon} \text{ and } \mu_e(E_{\sigma,\epsilon} - E) \leq \epsilon.$$

Proof. Let $E \subset X$ have finite outer measure.

Suppose E is μ-measurable.

Then $\mu(E) = \mu_e(E) < \infty$.

Since $\mu_e(E) < \infty$, for every $\epsilon > 0$ we have by Proposition 10.1 the existence of $E_{\sigma,\epsilon} \in \mathcal{Q}_\sigma \subset \mathcal{A}$ such that $E \subset E_{\sigma,\epsilon}$ and

$$\mu(E) + \epsilon = \mu_e(E) + \epsilon \geq \mu(E_{\sigma,\epsilon}).$$

Hence

$$-\mu(E) \leq -\mu(E_{\sigma,\epsilon}) + \epsilon.$$

Since μ is a measure on $\mathcal{A}$, we know for $A, B \in \mathcal{A}$ with $A \subset B$ that $\mu(B - A) = \mu(B) - \mu(A)$ when $\mu(A) < \infty$ (Proposition 3.1).

Thus with $A = E$ (hence $\mu(E) = \mu(A) < \infty$) and $B = E_{\sigma,\epsilon}$ we obtain

$$\mu(E_{\sigma,\epsilon} - E) = \mu(E_{\sigma,\epsilon}) - \mu(E) \leq \mu(E_{\sigma,\epsilon}) - \mu(E_{\sigma,\epsilon}) + \epsilon = \epsilon.$$

Since $E, E_{\sigma,\epsilon} \in \mathcal{A}$ we have $E_{\sigma,\epsilon} - E \in \mathcal{A}$, and since μ is the restriction of μ_e to $\mathcal{A}$, we obtain

$$\mu_e(E_{\sigma,\epsilon} - E) \leq \epsilon.$$

Now suppose that for every $\epsilon > 0$ there exists $E_{\sigma,\epsilon} \in \mathcal{Q}_\sigma$ such that $E \subset E_{\sigma,\epsilon}$ and $\mu_e(E_{\sigma,\epsilon} - E) \leq \epsilon$.

We are to verify that E is μ-measurable, i,e., for all $A \subset X$ there holds

$$\mu_e(A) \geq \mu_e(A \cap E) + \mu_e(A - E).$$

If $\mu_e(A) = \infty$, there is nothing to show, so suppose $\mu_e(A) < \infty$.

Since $E_{\sigma,\epsilon}$ is μ-measurable ($E_{\sigma,\epsilon} \in \mathcal{Q}_\sigma \subset \mathcal{A}$), $A \cap E_{\sigma,\epsilon} \supset A \cap E$, and $A - E_{\sigma,\epsilon} = (A - E) - A \cap (E_{\sigma,\epsilon} - E)$ (verify by Venn diagram), we have

$$\begin{aligned}\mu_e(A) &= \mu_e(A \cap E_{\sigma,\epsilon}) + \mu_e(A - E_{\sigma,\epsilon}) \\ &\geq \mu_e(A \cap E) + \mu_e\big((A - E) - A \cap (E_{\sigma,\epsilon} - E)\big).\end{aligned}$$

The outer measure μ_e is finitely subaddditive, so that as

$$A - E = \big((A - E) - A \cap (E_{\sigma,\epsilon} - E)\big) \cup (A \cap (E_{\sigma,\epsilon} - E))$$

(verify by Venn diagram), we have

$$\mu_e(A - E) \leq \mu_e\big((A - E) - A \cap (E_{\sigma,\epsilon} - E)\big) + \mu_e(A \cap (E_{\sigma,\epsilon} - E)).$$

Since $A \cap (E_{\sigma,\epsilon} - E) \subset A$ and μ_e is monotone, we have $\mu_e(A \cap (E_{\sigma,\epsilon} - E)) \leq \mu_e(A) < \infty$, so that

$$\mu_e(A) \geq \mu_e(A \cap E) + \mu_e(A - E) - \mu_e(A \cap (E_{\sigma,\epsilon} - E)).$$

Since $A \cap (E_{\sigma,\epsilon} - E) \subset E_{\sigma,\epsilon} - E$ and μ_e is monotone, we obtain

$$\mu_e(A \cap (E_{\sigma,\epsilon} - E)) \leq \mu_e(E_{\sigma,\epsilon} - E) \leq \epsilon,$$

and so

$$\mu_e(A) \geq \mu_e(A \cap E) + \mu_e(A - E) - \epsilon.$$

The arbitrariness of $\epsilon > 0$ gives the μ-measurability of E. □

Proposition 10.3. Let $\{X, \mathcal{A}, \mu\}$ be the measure space generated by the measure λ on the sequential covering and semialgebra $\mathcal{Q}$. A set $E \subset X$ of finite outer measure is μ-measurable if and only if there exists a set $E_{\sigma,\delta} \in \mathcal{Q}_{\sigma\delta}$ such that

$$E \subset E_{\sigma,\delta} \text{ and } \mu_e(E_{\sigma\delta} - E) = 0.$$

Proof. Let E have finite outer measure.

Suppose that E is μ-measurable.

Then $\mu(E) = \mu_e(E) < \infty$.

Since $\mu_e(E) < \infty$, there is by Proposition 10.1 a set $E_{\sigma\delta} \in \mathcal{Q}_{\sigma\delta}$ such that $E \subset E_{\sigma\delta}$ and $\mu_e(E) = \mu(E_{\sigma\delta})$.

The set $E_{\sigma\delta}$ is μ-measurable because $E_{\sigma\delta} \in \mathcal{Q}_{\sigma\delta} \subset \mathcal{A}$.

Since E is μ-measurable, the set $E_{\sigma\delta} - E$ is μ-measurable.

Because $\mu(E) < \infty$, we have

$$\mu_e(E_{\sigma\delta} - E) = \mu(E_{\sigma\delta} - E) = \mu(E_{\sigma\delta}) - \mu(E) = 0.$$

Now suppose there exists $E_{\sigma\delta} \in \mathcal{Q}_{\sigma\delta}$ such that $E \subset E_{\sigma\delta}$ and $\mu_e(E_{\sigma\delta} - E) = 0$.

Because $\mathcal{Q}_{\sigma\delta} \subset \mathcal{A}$, the set $E_{\sigma\delta}$ is μ-measurable.

We are to show that for all $A \in \mathcal{P}(X)$, there holds

$$\mu_e(A) \geq \mu_e(A \cap E) + \mu_e(A - E).$$

If $\mu_e(A) = \infty$, there is nothing to show, so we assume that $\mu_e(A) < \infty$.

Since $E_{\sigma\delta}$ is μ-measurable, $A \cap E_{\sigma\delta} \supset A \cap E$, and $A - E_{\sigma\delta} = (A - E) - A \cap (E_{\sigma\delta} - E)$ (verify by Venn diagram), we have

$$\begin{aligned} \mu_e(A) &= \mu_e(A \cap E_{\sigma\delta}) + \mu_e(A - E_{\sigma\delta}) \\ &\geq \mu_e(A \cap E) + \mu_e\big((A - E) - A \cap (E_{\sigma\delta} - E)\big) \end{aligned}$$

Since $A - E = \big((A - E) - A \cap (E_{\sigma\delta} - E)\big) \cup \big(A \cap (E_{\sigma\delta} - E)\big)$ (verify by Venn diagram) we have by the finite subadditivity of the outer measure that

$$\mu_e(A - E) \leq \mu_e\big((A - E) - A \cap (E_{\sigma\delta} - E)\big) + \mu_e\big(A \cap (E_{\sigma\delta} - E)\big).$$

Since $A \cap (E_{\sigma\delta} - E) \subset A$ and $\mu_e(A) < \infty$, we have by monotonicity of the outer measure that $\mu_e(A \cap (E_{\sigma\delta} - E)) < \infty$, so that

$$\mu_e((A - E) - A \cap (E_{\sigma\delta} - E)) \geq \mu_e(A - E) - \mu_e(A \cap (E_{\sigma\delta} - E)).$$

Thus we obtain

$$\mu_e(A) \geq \mu_e(A \cap E) + \mu_e(A - E) - \mu_e\big(A \cap (E_{\sigma\delta} - E)\big).$$

Since $A \cap (E_{\sigma\delta} - E) \subset E_{\sigma\delta} - E$, $\mu_e(E_{\sigma\delta} - E) = 0$, and μ_e is monotone, we obtain

$$\mu_e(A \cap (E_{\sigma\delta} - E)) = 0.$$

Therefore E is μ-measurable. $\square$