Math 541 Lecture #14
II.11: More on Extensions from Semialgebras to σ-algebras
II.12: The Lebesgue Measure of Sets in $\mathbb{R}^N$, Part I

11. More on Extensions from semiaglebras to σ-algebras. We will develop a bit more the theory on the extension of a measure λ on semialgebra and sequential covering $\mathcal{Q}$ to a measure space $\{X, \mathcal{A}, \mu\}$.

Theorem 11.1. Every measure λ on a semialgebra $\mathcal{Q}$ (and sequential covering of X) generates a measure space $\{X, \mathcal{A}, \mu\}$ where $\mathcal{A}$ is a σ-algebra containing $\mathcal{Q}$ and μ is a complete measure on $\mathcal{A}$, which agrees with λ on $\mathcal{Q}$. Moreover, if $\mathcal{Q}_0$ is the smallest σ-algebra containing $\mathcal{Q}$, then the restriction of μ to $\mathcal{Q}_0$ is an extension of λ to $\mathcal{Q}_0$, and this extension is unique if λ is a σ-finite measure (i.e., there is a countable $\{Q_n\}$ in $\mathcal{Q}$ such that $X = \cup Q_n$ and $\lambda(Q_n) < \infty$).

Proof. There only remains to show the uniqueness of the extension of λ to $\mathcal{Q}_0$ when λ is σ-finite.

To this end, suppose μ_1, μ_2 are extensions of λ to $\mathcal{Q}_0$, and let μ_e be the outer measure determined by λ on $\mathcal{Q}$.

The elements of $\mathcal{Q}_\sigma$ are elements of $\mathcal{Q}_0$, i.e., $\mathcal{Q}_\sigma \subset \mathcal{Q}_0$, because $\mathcal{Q}_0$ is the smallest σ-algebra containing $\mathcal{Q}$.

Every element of $\mathcal{Q}_\sigma$ can be written as the countable disjoint union of elements of $\mathcal{Q}$ (see the Lemma in Lecture #12).

Since the restriction of each of μ_1 and μ_2 to $\mathcal{Q}$ is λ, i.e., $\mu_i(Q) = \lambda(Q)$ for all $Q \in \mathcal{Q}$ (for each $i = 1, 2$), we have agreement of μ_1 and μ_2 on $\mathcal{Q}_\sigma$ by the countable additivity of the measures.

Since $\mu_e(Q) = \lambda(Q)$ for all $Q \in \mathcal{Q}$, we also have agreement of μ_1 and μ_2 with μ_e on $\mathcal{Q}_\sigma$.

Next, for $E \in \mathcal{Q}_0$ having finite outer measure $\mu_e(E)$, we will show that $\mu_1(E)$ and $\mu_2(E)$ agree with $\mu_e(E)$.

For every $\epsilon > 0$ there is by Proposition 10.1 (applied to μ_1 as an extension of λ) a set $E_{\sigma,\epsilon} \in \mathcal{Q}_\sigma$ such that $E \subset E_{\sigma,\epsilon}$ and

$$\mu_1(E_{\sigma,\epsilon}) \leq \mu_e(E) + \epsilon.$$

These imply by the monotonicity of μ_1 that

$$\mu_1(E) \leq \mu_1(E_{\sigma,\epsilon}) \leq \mu_e(E) + \epsilon.$$

We will prove the opposite inequality.

Since $\epsilon > 0$ is arbitrary, we obtain for all $E \in \mathcal{Q}_0$ of finite outer measure that

$$\mu_1(E) \leq \mu_e(E).$$

Since $E \in \mathcal{Q}_0$ and $E_{\sigma,\epsilon} \in \mathcal{Q}_\sigma \subset \mathcal{Q}_0$, we have $E_{\sigma,\epsilon} - E \in \mathcal{Q}_0$.

The restriction of μ_e to $\mathcal{Q}_0$ is a measure because the restriction of μ_e to $\mathcal{A}$ is a measure and $\mathcal{Q}_0 \subset \mathcal{A}$.

Then, since $\mu_e(E) < \infty$, $\mu_1 = \mu_e$ on $\mathcal{Q}_\sigma$, and $\mu_1(E_{\sigma,\epsilon}) - \mu_e(E) \leq \epsilon$, we have

$$\mu_e(E_{\sigma,\epsilon} - E) = \mu_e(E_{\sigma,\epsilon}) - \mu_e(E) = \mu_1(E_{\sigma,\epsilon}) - \mu_e(E) \leq \epsilon.$$

Since $E \subset E_{\sigma,\epsilon}$, we have $E_{\sigma\epsilon} = E \cup (E_{\sigma,\epsilon} - E)$ as a disjoint union of elements of $\mathcal{Q}_0$.

Since μ_e and μ_1 are measures on $\mathcal{Q}_0$ with $\mu_e = \mu_1$ on $\mathcal{Q}_\sigma$, $\mu_1(F) \leq \mu_e(F)$ for all $F \in \mathcal{Q}_0$ satisfying $\mu_e(F) < \infty$, and $\mu_e(E_{\sigma,\epsilon} - E) \leq \epsilon$, we have

$$\begin{aligned} \mu_e(E) &\leq \mu_e(E_{\sigma,\epsilon}) = \mu_1(E_{\sigma,\epsilon}) \\ &= \mu_1(E) + \mu_1(E_{\sigma,\epsilon} - E) \\ &\leq \mu_1(E) + \mu_e(E_{\sigma,\epsilon} - E) \\ &\leq \mu_1(E) + \epsilon. \end{aligned}$$

The arbitrariness of ϵ implies that $\mu_e(E) \leq \mu_1(E)$.

Thus for $E \in \mathcal{Q}_0$ with finite outer measure we obtain $\mu_1(E) = \mu_e(E)$.

Replacing μ_1 with μ_2 in this argument results in $\mu_2(E) = \mu_e(E)$ for all $E \in \mathcal{Q}_0$ with finite outer measure.

Finally, we show that $\mu_1(E) = \mu_2(E)$ for any $E \in \mathcal{Q}_0$.

Since λ is σ-finite, there is a countably collection $\{Q_n\}$ in $\mathcal{Q}$ such that $\mu_e(Q_n) < \infty$ for each n, and

$$E = \bigcup_{n=1}^{\infty} Q_n \cap E,$$

where each $Q_n \cap E$ is in $\mathcal{Q}_0$ and has finite outer measure.

As done before, we may assume that the collection $\{Q_n\}$ is pairwise disjoint.

Since μ_1 and μ_2 agree on those elements of $\mathcal{Q}_0$ of finite outer measure, we have

$$\mu_1(E) = \sum_{n=1}^{\infty} \mu_1(Q_n \cap E) = \sum_{n=1}^{\infty} \mu_2(Q_n \cap E) = \mu_2(E).$$

This shows that $\mu_1 = \mu_2$ on $\mathcal{Q}_0$. □

The lack of uniqueness of the extension for non-σ-finite measures λ is a homework problem.

12. The Lebesgue Measure of Sets in $\mathbb{R}^N$. The collection $\mathcal{Q}$ of $\frac{1}{2}$-closed dyadic cubes, including the empty set, is a semialgebra and a sequential covering of $\mathbb{R}^N$.

The Euclidean measure (or volume) λ of the $\frac{1}{2}$-closed dyadic cubes in $\mathbb{R}^N$ is a σ-finite measure on $\mathcal{Q}$, and hence λ is finitely additive and countably subadditive.

The measure λ on $\mathcal{Q}$ determines an outer measure μ_e on $\mathbb{R}^N$, called the **Lebesgue outer measure** on $\mathbb{R}^N$.

Proposition 12.1. Let $\mathcal{M}$ be the σ-algebra generated by the Euclidean measure λ on $\mathcal{Q}$. Then $\mathcal{M}$ contains

(i) the $\frac{1}{2}$-closed dyadic cubes $\mathcal{Q}$,
(ii) all open subsets of $\mathbb{R}^N$,
(iii) all closed subsets of $\mathbb{R}^N$, and
(iv) all sets in $\mathbb{R}^N$ of the form $\mathcal{F}_\sigma$, $\mathcal{G}_\delta$, $\mathcal{F}_{\sigma\delta}$, $\mathcal{G}_{\delta\sigma}$, etc.

Proof. The collection of $\frac{1}{2}$-closed dyadic cubes $\mathcal{Q}$ and the Euclidean measure λ on them satisfy the assumptions of Proposition 9.1.

This implies that $\mathcal{Q} \subset \mathcal{M}$.

Since every open set in $\mathbb{R}^N$ is a countable union of $\frac{1}{2}$-closed dyadic cubes, the open sets are in $\mathcal{M}$.

Since the σ-algebra $\mathcal{M}$ is closed under complements, every closed subset is in $\mathcal{M}$.

Since the σ-algebra $\mathcal{M}$ is closed under countable unions and countable intersections, sets of the type $\mathcal{F}_\sigma$, $\mathcal{G}_\delta$, etc., are in $\mathcal{M}$. □

The restriction μ of the Lebesgue outer measure μ_e to $\mathcal{M}$ is the **Lebesgue measure** in $\mathbb{R}^N$, and the sets in $\mathcal{M}$ are called the **Lebesgue measurable** sets in $\mathbb{R}^N$.

The Lebesgue measure μ on $\mathcal{M}$ is an extension of the Euclidean measure λ on $\mathbb{Q}$, that is μ extends the notion of volume from dyadic cubes to the sets in $\mathcal{M}$.

Recall the the Borel σ-algebra $\mathcal{B}$ is the smallest σ-algebra containing the open sets.

Also, each element of the Borel σ-algebra is called a Borel set.

The Borel sets include the sets of type $\mathcal{F}_\sigma$, $\mathcal{G}_\delta$, etc.

Corollary. (i) $\mathcal{B} \subset \mathcal{M}$. (ii) Every finite or countable infinite subset of $\mathbb{R}^N$ has Lebesgue measure 0.

Proof. (i) Because $\mathcal{M}$ is σ-algebra containing the open sets and $\mathcal{B}$ is the smallest σ-algebra containing the open sets, we have $\mathcal{B} \subset \mathcal{M}$.

(ii) For $x \in \mathbb{R}^N$, the singleton set $\{x\}$ belongs to $\mathcal{M}$ because its complement is open.

Since a singleton set $\{x\}$ belongs to $\frac{1}{2}$-closed dyadic cubes of arbitrarily small Euclidean measure, we have $\mu_e(\{x\}) = 0$, and so $\mu(\{x\}) = 0$ since μ is the restriction of μ_e to $\mathcal{M}$.

A finite subset of $\mathbb{R}^N$ is a disjoint union of singleton sets.

Since each singleton set has Lebesgue measure zero, their finite union has Lebesgue measure zero by finite additivity.

A countably infinite subset of $\mathbb{R}^N$ is a disjoint union of singleton sets.

Since each singleton set has Lebesgue measure zero, their countable union has Lebesgue measure zero by countable additivity. □

Remark. We will see soon that there are Lebesgue measurable sets that are not Borel sets, that is, the inclusion $\mathcal{B} \subset \mathcal{M}$ is strict.

Homework Problem 14A. Let $\mathcal{Q}_0$ be the smallest σ-algebra containing the semialgebra $\mathcal{Q}$ of $\frac{1}{2}$-closed dyadic cubes in $\mathbb{R}^N$. Prove that $\mathcal{Q}_0$ is the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}^N$.