Math 541 Lecture #15
II.12: The Lebesgue Measure of Sets in $\mathbb{R}^N$, Part II

§12.1: A Necessary and Sufficient Condition of Measurability. We will apply the necessary and sufficient conditions we obtained before for the measurability of sets to Lebesgue measure and subsets of $\mathbb{R}^N$.

Let μ be the Lebesgue measure in $\mathbb{R}^N$, and for any subset E of $\mathbb{R}^N$, define

$$\mu_e'(E) = \inf\{\mu(\mathcal{O}) : \mathcal{O} \text{ is open}, E \subset \mathcal{O}\}.$$

This function μ_e' is a bit like the outer measure μ_e from which Lebesgue was generated, except the class of sets over which the infimum is taken is different.

Lemma. The set function μ_e' satisfies $\mu_e(E) \leq \mu_e'(E)$ for all $E \in \mathcal{P}(\mathbb{R}^N)$.

Proof. Recall that

$$\mu_e(E) = \inf\left\{\sum_{n=1}^{\infty} \lambda(Q_n) : E \subset \bigcup_{n=1}^{\infty} Q_n, Q_n \in \mathcal{Q}\right\}.$$

Since $\mu = \lambda$ on $\mathcal{Q}$, we have

$$\mu_e(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(Q_n) : E \subset \bigcup_{n=1}^{\infty} Q_n, Q_n \in \mathcal{Q}\right\}.$$

On the other hand, since every open set $\mathcal{O}$ is the pairwise disjoint union of $\frac{1}{2}$-closed dyadic cubes $\{Q_n\} \subset \mathcal{Q}$, we have

$$\begin{aligned}\mu_e'(E) &= \inf\left\{\mu(\mathcal{O}) : E \subset \mathcal{O} = \bigcup_{n=1}^{\infty} Q_n, Q_n \in \mathcal{Q}\right\}\\ &= \inf\left\{\sum_{n=1}^{\infty} \mu(Q_n) : E \subset O = \bigcup_{n=1}^{\infty} Q_n, Q_n \in \mathcal{Q}\right\}\end{aligned}$$

by countable additivity of μ.

The set over which the inf for $\mu_e(E)$ is taken contains the set over which the inf for $\mu_e'(E)$ is taken.

Therefore $\mu_e(E) \leq \mu_e'(E)$. □

Note that every closed cube in $\mathbb{R}^N$ (not just the dyadic cubes) and their interiors are Lebesgue measurable because the closed cubes are closed subsets and the interiors are open subsets.

Proposition 12.2. If a subset E of $\mathbb{R}^N$ has finite outer measure, then $\mu_e(E) = \mu_e'(E)$.

Proof. Suppose the outer measure of E is finite.

By observation (2), we need only show that $\mu_e'(E) \leq \mu_e(E)$.

For $\epsilon > 0$, let $\{\mathcal{Q}_{\epsilon,n}\}$ be a countable collection of $\frac{1}{2}$-closed dyadic cubes such that

$$E \subset \bigcup_{n=1}^{\infty} \mathcal{Q}_{\epsilon,n}, \quad \sum_{n=1}^{\infty} \mu(\mathcal{Q}_{\epsilon,n}) \le \mu_e(E) + \epsilon.$$

For each n there exists a closed cube $\mathcal{Q}'_{\epsilon,n}$ (not necessarily dyadic) such that

$$\mathcal{Q}_{\epsilon,n} \subset \mathring{\mathcal{Q}}'_{\epsilon,n}, \;\; \mu(\mathring{\mathcal{Q}}'_{\epsilon,n} - \mathcal{Q}_{\epsilon,n}) \le \frac{\epsilon}{2^n}.$$

The union of the $\mathring{\mathcal{Q}}'_{\epsilon,n}$ is open and contains E, so that

$$\begin{aligned} \mu'_e(E) &\le \mu\left(\bigcup \mathring{\mathcal{Q}}'_{\epsilon,n}\right) \le \sum \mu(\mathring{\mathcal{Q}}'_{\epsilon,n}) \\ &= \sum \mu\left(\mathcal{Q}_{\epsilon,n} \cup (\mathring{\mathcal{Q}}'_{\epsilon,n} - \mathcal{Q}_{\epsilon,n})\right) \\ &= \sum \left(\mu(\mathcal{Q}_{\epsilon,n}) + \mu(\mathring{\mathcal{Q}}'_{\epsilon,n} - \mathcal{Q}_{\epsilon,n})\right) \\ &\le \mu_e(E) + \epsilon + \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} \\ &\le \mu_e(E) + 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain $\mu'_e(E) \le \mu_e(E)$. □

Proposition 12.3. A subset E of $\mathbb{R}^N$ of finite outer measure is Lebesgue measurable if and only if for every $\epsilon > 0$ there exists an open set $E_{o,\epsilon}$ such that

$$E \subset E_{o,\epsilon} \text{ and } \mu_e(E_{o,\epsilon} - E) \le \epsilon.$$

Moreover, any set E of finite outer measure is Lebesgue measurable if and only if there is a set E_δ of type $\mathcal{G}_\delta$ such that

$$E \subset E_\delta \text{ and } \mu_e(E_\delta - E) = 0.$$

Proof. Suppose a set E of finite outer measure is Lebesgue measurable.

Then for every $\epsilon > 0$ there is, by Proposition 10.2, a set $E_{\sigma,\epsilon}$ in $\mathcal{Q}_\sigma$ such that

$$E \subset E_{\sigma,\epsilon} \text{ and } \mu_e(E_{\sigma,\epsilon} - E) \le \frac{\epsilon}{2}.$$

Since $E_{\sigma,\epsilon}$ and E are Lebesgue measurable we have that $E_{\sigma,\epsilon} - E$ is Lebesgue measurable.

Since $\mu(E) = \mu_e(E) < \infty$, we have

$$\mu_e(E_{\sigma,\epsilon}) - \mu_e(E) = \mu(E_{\sigma,\epsilon}) - \mu(E) = \mu(E_{\sigma,\epsilon} - E) = \mu_e(E_{\sigma,\epsilon} - E) \le \frac{\epsilon}{2}.$$

Thus $\mu_e(E_{\sigma,\epsilon}) < \infty$, so that by Proposition 12.2 we have $\mu_e(E_{\sigma,\epsilon}) = \mu'_e(E_{\sigma,\epsilon})$.

Hence there is an open set $E_{o,\epsilon}$ such that $E_{\sigma,\epsilon} \subset E_{o,\epsilon}$ and

$$\mu(E_{o,\epsilon}) \leq \mu_e(E_{\sigma,\epsilon}) + \frac{\epsilon}{2}.$$

These imply that $E \subset E_{o,\epsilon}$ and (by subtracting the finite $\mu_e(E)$ from both sides)

$$\mu(E_{o,\epsilon}) - \mu_e(E) \leq \mu_e(E_{\sigma,\epsilon}) - \mu_e(E) + \frac{\epsilon}{2} \leq \epsilon.$$

Since E and $E_{o,\epsilon}$ are Lebesgue measurable with $\mu(E) = \mu_e(E) < \infty$, we have

$$\mu_e(E_{o,\epsilon} - E) = \mu(E_{o,\epsilon} - E) = \mu(E_{o,\epsilon}) - \mu(E) = \mu(E_{o,\epsilon}) - \mu_e(E) \leq \epsilon.$$

Now for a set E of finite outer measure, suppose for every $\epsilon > 0$ there exists an open $E_{o,\epsilon}$ such that $E \subset E_{o,\epsilon}$ and

$$\mu_e(E_{o,\epsilon} - E) \leq \epsilon.$$

Since every open set is the countable union of $\frac{1}{2}$-closed dyadic cubes, we have $E_{o,\epsilon} \in \mathcal{Q}_\sigma$.

Hence by Proposition 10.2, we have that E is Lebesgue measurable.

Now suppose that a set E of finite outer measure is Lebesgue measurable.

By the above argument, there is for each $n \in \mathbb{N}$ an open set $E_{o,1/n}$ such that

$$E \subset E_{o,1/n} \text{ and } \mu_e(E_{o,1/n}) \leq \mu_e(E) + \frac{1}{n}.$$

Since $E \subset \cap_{k=1}^\infty E_{o,1/k} \subset E_{o,1/n}$ for all n, we have

$$\mu_e(E) \leq \mu_e(\cap E_{o,1/k}) \leq \mu_e(E_{o,1/n}) \leq \mu_e(E) + \frac{1}{n}.$$

For $E_\delta = \cap E_{o,1/n}$, we obtain as $n \to \infty$ that

$$\mu_e(E) - \mu_e(E_\delta) = 0.$$

Since E and E_δ are Lebesgue measurable with $\mu(E) = \mu_e(E) < \infty$ we have

$$\mu_e(E_\delta - E) = \mu(E_\delta - E) = \mu(E_\delta) - \mu(E) = \mu_e(E_\delta) - \mu_e(E) = 0.$$

Finally suppose for a set E of finite outer measure there exists a set E_δ of type $\mathcal{G}_\delta$ such that

$$E \subset E_\delta \text{ and } \mu_e(E_\delta - E) = 0.$$

Since each open set is a countable union of $\frac{1}{2}$-closed dyadic cubes in $\mathcal{Q}$, the set E_δ belongs to $\mathcal{Q}_{\sigma\delta}$.

Thus by Proposition 10.3, the set E is Lebesgue measurable. □