Math 541 Lecture #16
II.12: The Lebesgue Measure of Sets in $\mathbb{R}^N$, Part III

12: The Lebesgue measure of sets in $\mathbb{R}^N$. We continue with characterizing the Lebesgue measurable sets in $\mathbb{R}^N$.

Proposition 12.4. A bounded set $E \subset \mathbb{R}^N$ is Lebesgue measurable if and only if for every $\epsilon > 0$ there exists a closed set $E_{c,\epsilon}$ such that

$$E_{c,\epsilon} \subset E \text{ and } \mu_e(E - E_{c,\epsilon}) \leq \epsilon.$$

Moreover, a bounded set E is Lebesgue measurable if and only if there exists a set E_σ of type $\mathcal{F}_\sigma$ type such that

$$E_\sigma \subset E \text{ and } \mu_e(E - E_\sigma) = 0.$$

Proof. Suppose that a bounded E is Lebesgue measurable.

Then there is a closed cube Q such that $E \subset Q$ and $Q - E$ has finite outer measure.

Thus for every $\epsilon > 0$ there is by Proposition 12.3 the existence of an open set $E_{o,\epsilon}$ such that

$$(Q - E) \subset E_{o,\epsilon} \text{ and } \mu_e(E_{o,\epsilon} - (Q - E)) \leq \epsilon.$$

The set $E_{c,\epsilon} = E_{o,\epsilon}^c \cap Q$ is closed (bounded and hence compact) and satisfies

$$\begin{aligned} E - E_{c,\epsilon} &= E - (E_{o,\epsilon}^c \cap Q) = E \cap (E_{o,\epsilon}^c \cap Q)^c \\ &= E \cap (E_{o,\epsilon} \cup Q^c) = (E \cap E_{o,\epsilon}) \cup (E \cap Q^c) \\ &= (E \cap E_{o,\epsilon}) \cup \emptyset = E \cap E_{o,\epsilon}. \end{aligned}$$

Since

$$\begin{aligned} E_{o,\epsilon} - (Q - E) &= E_{o,\epsilon} \cap (Q - E)^c = E_{o,\epsilon} \cap (Q \cap E^c)^c \\ &= E_{o,\epsilon} \cap (Q^c \cup E) = (E_{o,\epsilon} \cap Q^c) \cup (E_{o,\epsilon} \cap E) \end{aligned}$$

we obtain that

$$E - E_{c,\epsilon} \subset E_{0,\epsilon} - (Q - E).$$

Thus by monotonicity of the outer measure we have

$$\mu_e(E - E_{c,\epsilon}) \leq \mu_e(E_{o,\epsilon} - (Q - E)) \leq \epsilon.$$

Now suppose that a bounded set E has for every $\epsilon > 0$ the existence of a closed set $E_{c,\epsilon}$ such that

$$E_{c,\epsilon} \subset E \text{ and } \mu_e(E - E_{c,\epsilon}) \leq \frac{\epsilon}{2}.$$

By Proposition 12.2, we have $\mu_e(E - E_{c,\epsilon}) = \mu_e'(E - E_{c,\epsilon})$, and so there exists an open set $E_{o,\epsilon}$ (which is Lebesgue measurable) such that

$$E - E_{c,\epsilon} \subset E_{o,\epsilon} \text{ and } \mu(E_{o,\epsilon}) = \mu_e(E_{o,\epsilon}) \leq \mu_e(E - E_{c,\epsilon}) + \frac{\epsilon}{2}.$$

Since $\mu_e(E - E_{c,\epsilon}) \leq \epsilon/2$ we obtain $\mu_e(E_{o,\epsilon}) \leq \epsilon$.

Since $E_{o,\epsilon} - (E - E_{c,\epsilon}) \subset E_{o,\epsilon}$, we have by monotonicity of the outer measure that

$$\mu_e(E_{o,\epsilon} - (E - E_{c,\epsilon})) \leq \mu_e(E_{o,\epsilon}) \leq \epsilon.$$

This shows by Proposition 12.3 that $E - E_{c,\epsilon}$ is Lebesgue measurable.

Since $E_{c,\epsilon} \subset E$ we have $E = (E - E_{c,\epsilon}) \cup E_{c,\epsilon}$ (disjointly).

Since $E - E_{c,\epsilon}$ and $E_{c,\epsilon}$ are Lebesgue measurable, their union E is also.

Now suppose that a bounded set E is Lebesgue measurable.

By the argument above there is for each $n \in \mathbb{N}$ a closed set $E_{c,1/n}$ such that

$$E_{c,1/n} \subset E \text{ and } \mu_e(E - E_{c,1/n}) \leq \frac{1}{n}.$$

Since E and $E_{c,1/n}$ are Lebesgue measurable, so is $E - E_{c,1/n}$.

Since E is a bounded subset, it has finite outer measure, and hence finite Lebesgue measure, so that $\mu(E_{c,1/n}) < \infty$, and hence

$$\mu_e(E - E_{c,1/n}) = \mu(E - E_{c,1/n}) = \mu(E) - \mu(E_{c,1/n}) = \mu_e(E) - \mu_e(E_{c,1/n}).$$

We obtain

$$\frac{1}{n} \geq \mu_e(E - E_{c,1/n}) = \mu_e(E) - \mu_e(E_{c,1/n}).$$

Rewritten, this is

$$\mu_e(E_{c,1/n}) \geq \mu_e(E) - \frac{1}{n}.$$

The set $E_\sigma = \cup E_{c,1/n}$ is of type $\mathcal{F}_\sigma$ and satisfies $E_{c,1/n} \subset E_\sigma \subset E$ for all n.

Thus by the monotonicity of μ_e we have for all n that

$$-\frac{1}{n} + \mu_e(E) \leq \mu_e(E_{c,1/n}) \leq \mu_e(E_\sigma) \leq \mu_e(E).$$

This implies by the Squeeze Theorem that $\mu_e(E) = \mu_e(E_\sigma)$ as $n \to \infty$.

Since E and E_σ are Lebesgue measurable with $\mu(E) < \infty$ and hence $\mu(E_\sigma) < \infty$, we have

$$\mu_e(E - E_\sigma) = \mu(E - E_\sigma) = \mu(E) - \mu(E_\sigma) = \mu_e(E) - \mu_e(E_\sigma) = 0.$$

Finally, suppose for a bounded set E that there exists a set E_σ of type $\mathcal{F}_\sigma$ such that $E_\sigma \subset E$ and $\mu_e(E - E_\sigma) = 0$.

Since E is bounded, the set $E - E_\sigma$ has finite outer measure.

By Proposition 12.3, there exists a set E_δ of type $\mathcal{G}_\delta$ that $E - E_\sigma \subset E_\delta$ and

$$\mu_e(E_\delta - (E - E_\sigma)) = 0.$$

Since each open set is the countable union of $\frac{1}{2}$-closed dyadic cubes, the set E_δ belongs to $\mathcal{Q}_{\sigma\delta}$.

The set $E - E_\sigma$ is then Lebesgue measurable by Proposition 10.3.

Since $E_\sigma \subset E$ we have $E = (E - E_\sigma) \cup E_\sigma$ (disjointly).

Since E_σ is Lebesgue measurable, we arrive at E being Lebesgue measurable. $\square$