Math 541 Lecture #17
II.13: A Nonmeasurable Set, Part I

13: Existence of Nonmeasurable Subset of $[0,1)$. We exhibit the existence of a subset of the interval $[0,1)$ which is not Lebesgue measurable.

Define $\bullet : [0,1) \times [0,1) \to [0,1)$ by

$$x \bullet y = \begin{cases} x+y & \text{if } x+y<1, \\ x+y-1 & \text{if } x+y \geq 1. \end{cases}$$

[The set $[0,1)$ with this binary operation is isomorphic to the group S^1.]

For a subset E of $[0,1)$ and $y \in [0,1)$ define

$$E \bullet y = \{x \bullet y : x \in E\}.$$

Lemma. If A is a Lebesgue measurable subset of $[0,1)$ and $y \in [0,1)$, then $A \bullet y$ is Lebesgue measurable and $\mu(A \bullet y) = \mu(A)$.

Homework Problem 17A. Give a proof of this Lemma.

We define an equivalence relation $\sim$ in $[0,1)$ by

$$x \sim y \text{ if } x - y \in \mathbb{Q}.$$

If $\mathcal{E}$ is an equivalence class for this equivalence relation, then two elements of $\mathcal{E}$ differ by a rational number.

The set $\mathbb{Q} \cap [0,1)$ is one such equivalence class.

Using the Axiom of Choice, we select a subset E of $[0,1)$ that contains one and only one element of each of the equivalence classes.

We will show that E is not Lebesgue measurable.

Two distinct elements of E have the property that they are not equivalent, i.e., they do not differ by a rational number.

Set $r_0 = 0$ and let r_n be an enumeration of the elements of $\mathbb{Q} \cap (0,1)$.

For each $n = 0,1,2,3,\ldots$, form the sets

$$E_n = E \bullet r_n.$$

Claim 1. The sets $\{E_n\}$ are pairwise disjoint.

Suppose for n,m that $E_n \cap E_m \neq \emptyset$, and let $x \in E_n \cap E_m$.

Then there are elements $x_n, x_m \in E$ for which $x_n \bullet r_n = x_m \bullet r_m$.

There are four similar cases to consider depending on values of $x_n + r_n$ and $x_m + r_m$.

If $x_n + r_n < 1$ and $x_m + r_m < 1$, then $x_n \bullet r_n = x_m \bullet r_m$ implies

$$x_n + r_n = x_m + r_m.$$

If $x_n + r_n \geq 1$ and $x_m + r_m < 1$, then $x_n \bullet r_n = x_m \bullet r_m$ implies

$$x_n + r_n - 1 = x_m + r_m.$$

If $x_n + r_m < 1$ and $x_m + r_m \geq 1$, then $x_n \bullet r_n = x_m \bullet r_m$ implies

$$x_n + r_n = x_m + r_m - 1.$$

If $x_n + r_m \geq 1$ and $x_m + r_m \geq 1$, then $x_n \bullet r_n = x_m \bullet r_m$ implies

$$x_n + r_n - 1 = x_m + r_m - 1.$$

In all four cases we get $x_n - x_m \in \mathbb{Q}$.

But no two distinct elements of E differ by a rational number.

So $x_n = x_m$, and the equation $x_n \bullet r_n = x_m \bullet r_m$ gives four cases (as above) of

$$\begin{aligned} r_n &= r_m, \\ r_n - 1 &= r_m, \\ r_n &= r_m - 1, \\ r_n &= r_m. \end{aligned}$$

In the two middle cases, since $r_n < 1$ and $r_m < 1$ we have $r_m = r_n - 1 < 0$ or $r_n = r_m - 1 < 0$ both of which are impossible, so that $r_n = r_m$.

Hence $n = m$ so that $E_n = E_m$, giving the Claim.

Claim 2. Each element of $[0, 1)$ belongs to E_n for some n.

Every $x \in [0, 1)$ belongs to some equivalence class, so there exists $y \in E$ such that $x - y \in \mathbb{Q}$.

If $x - y \geq 0$, then $x = y + r_n$ for some r_n, and hence $x \in E_n$.

If $x - y < 0$, then since $x, y \in [0, 1)$ there holds $-1 < x - y$, so that for some r_n with $n \geq 1$ (i.e., $r_n \neq 0$) we have $x - y = -r_n$; this means

$$y + (1 - r_n) = (y - r_n) + 1 = x + 1 \geq 1,$$

and hence

$$x = y - r_n = y + (1 - r_n) - 1 = y \bullet (1 - r_n).$$

Since $r_n \neq 0$, there exists $m \in \mathbb{N}$ such that $1 - r_n = r_m$.

Thus $x = y \bullet r_m$ and so $x \in E_m$.

In either case we have $x \in E_n$ for some n, giving the Claim.

By Claim 2 we have

$$[0, 1) = \bigcup E_n.$$

If E were Lebesgue measurable, then by the Lemma, each $E_n = E \bullet r_n$ would be Lebesgue measurable and satisfy $\mu(E_n) = \mu(E \bullet r_n) = \mu(E)$.

By Claim 1, the sets $\{E_n\}$ are pairwise disjoint, so by countable additivity we have

$$\mu([0,1)) = \sum_{n=0}^{\infty} \mu(E_n) = \sum_{n=0}^{\infty} \mu(E).$$

The value of $\sum \mu(E)$ is either 0 or ∞.

But the value of $\mu([0,1))$ is 1 because in terms of the Lebesgue measurable singleton sets and $\frac{1}{2}$-closed dyadic intervals we have

$$[0,1) = \big(\{0\} \cup (0,1/2] \cup (1/2,1]\big) - \{1\},$$

whence $[0,1) \in \mathcal{M}$ and as the Lebesgue measure of singleton sets is zero,

$$\begin{aligned}
\mu([0,1)) &= \mu(\{0\}) + \mu((0,1/2]) + \mu((1/2,1]) - \mu(\{1\}) \\
&= \lambda((0,1/2]) + \lambda((1/2,1]) \\
&= 1/2 + 1/2 = 1.
\end{aligned}$$

This contradiction implies that E is not Lebesgue measurable.

The set E is called a Vitali nonmeasurable set.

There are many ways to select the elements that constitute E, and hence there are many nonmeasurable subsets of $[0,1)$.