Math 541 Lecture #18
II.13: A Nonmeasurable Set, Part II

13: Existence of a Nonmeasurable set in $\mathbb{R}^N$. We make use of the subgroup of rational points $\mathbb{Q}^N$ inside the group $\mathbb{R}^N$.

The cosets of the quotient group

$$\mathbb{R}^N/\mathbb{Q}^N = \{x + \mathbb{Q}^N : x \in \mathbb{R}^N\}$$

form a cover of $\mathbb{R}^N$, i.e., their union is all of $\mathbb{R}^N$.

Furthermore, for any $x, x' \in \mathbb{R}^N$, there holds $y \in x + \mathbb{Q}^N$ if and only if $y - x \in \mathbb{Q}^N$, and $x + \mathbb{Q}^N = x' + \mathbb{Q}^N$ if and only if $x - x' \in \mathbb{Q}^N$.

Claim 1. Two cosets of $\mathbb{R}^N/\mathbb{Q}^N$ are either identical or disjoint, that is, for $x, x' \in \mathbb{R}^N$, either $x + \mathbb{Q}^N = x' + \mathbb{Q}^N$ or

$$(x + \mathbb{Q}^N) \cap (x' + \mathbb{Q}^N) = \emptyset.$$

Suppose that $(x + \mathbb{Q}^N) \cap (x' + \mathbb{Q}^N) \neq \emptyset$.

Then there exists y such $y \in x + \mathbb{Q}^N$ and $y \in x' + \mathbb{Q}^N$.

Hence there exist $r, r' \in \mathbb{Q}^N$ such that $y = x + r$ and $y = x' + r'$.

Thus $x + r = x' + r'$, or $x - x' = r' - r \in \mathbb{Q}^N$.

This implies $x + \mathbb{Q}^N = x' + \mathbb{Q}^N$, and gives the Claim.

Since the union of the cosets is all of $\mathbb{R}^N$, each point of $\mathbb{R}^N$ belongs to precisely one coset $x + \mathbb{Q}^N$.

By the Axiom of Choice, we choose exactly one point from each coset of $\mathbb{R}^N/\mathbb{Q}^N$.

Let E be the collection of these points.

By Claim 1 we then have the pairwise disjoint union

$$\mathbb{R}^N = \bigcup_{x \in E} \big(x + \mathbb{Q}^N\big).$$

This says that for each $y \in \mathbb{R}^N$ there exists a unique $z \in E$ and a unique $r \in \mathbb{Q}^N$ such that $y = r + z$.

By enumerating $\mathbb{Q}^N = \{r_1, r_2, r_3, \dots\}$ we obtain the union

$$\mathbb{R}^N = \bigcup_{k=1}^{\infty} \big(r_k + E\big).$$

Claim 2. This union is a pairwise disjoint union.

For $k, l \in \mathbb{N}$ we have either $(r_k + E) \cap (r_l + E) = \emptyset$ or $(r_k + E) \cap (r_l + E) \neq \emptyset$.

In the latter case, there is $y \in (r_k + E) \cap (r_l + E)$, and so $y = r_k + z_1$ and $y = r_l + z_2$ for some $z_1, z_2 \in E$.

Hence $z_1 - z_2 = (y - r_k) - (y - r_l) = r_l - r_k \in \mathbb{Q}^n$, meaning that $z_1 + \mathbb{Q}^N = z_2 + \mathbb{Q}^N$.

By the way the points of E were chosen, we must have $z_1 = z_2 = z$.

Then $r_k + z = y = r_l + z$ implies $r_k = r_l$ and hence $k = l$.

Thus $r_k + E = r_l + E$, and this gives the Claim.

If $\mu_e(E) = 0$, then by part (iii) of Proposition 6.1, the set E would be a Lebesgue measurable set of Lebesgue measure 0, implying by translation invariance that $\mu(r_k + E) = 0$ for all k, hence by countable additivity that

$$\infty = \mu(\mathbb{R}^N) = \sum_{k=1}^{\infty} \mu(r_k + E) = \sum_{k=1}^{\infty} \mu(E) = 0.$$

By this contradiction, we obtain $\mu_e(E) > 0$.

Now let K be a compact subset of E. [Such do exist, the singleton subsets of the nonempty E being compact.]

For the bounded, countably infinite set $D = B_1(0) \cap \mathbb{Q}^N$, we have the union

$$\bigcup_{r \in D} (r + K)$$

is a bounded set in $\mathbb{R}^N$, and hence has finite Lebesgue outer measure.

This union is pairwise disjoint because $r + K \subset r + E$ and the collection of sets of the form $r + E$ are pairwise disjoint by Claim 2.

Since K is compact, it is closed, and hence Lebesgue measurable, so that by translation invariance each $r + K$ is Lebesgue measurable, their union is Lebesgue measurable, and $\mu(r + K) = \mu(K)$ for all $r \in D$.

Thus by countable additivity of Lebesgue measure we have

$$\infty > \mu\left(\bigcup_{r \in D} (r + K)\right) = \sum_{r \in D} \mu(r + K) = \sum_{r \in D} \mu(K).$$

This implies that $\mu(K) = 0$ for all compact subsets K of E.

While it is true that E is not Lebesgue measurable, we have not developed the tools needed to show this for the possibly unbounded set E.

Instead we consider for some integer $l \geq 1$ the bounded set $E_l = E \cap B_l(0)$.

If $\mu_e(E_l) = 0$ for all l, then by subadditivity of the Lebesgue outer measure we would have

$$\mu_e(E) = \mu_e\left(\bigcup_{l \in \mathbb{N}} E_l\right) \leq \sum_{l=1}^{\infty} \mu_e(E_l) = 0,$$

contradicting $\mu_e(E) > 0$.

So there exists $l \in \mathbb{N}$ such that $\mu_e(E_l) > 0$.

Suppose E_l is Lebesgue measurable. Then $\mu(E_l) = \mu_e(E_l) > 0$.

Since E_l is bounded, there is by Proposition 12.4 an $\mathcal{F}_\sigma$ set F such that $F \subset E_l$ and

$$\mu_e(E_l - F) = 0.$$

Since F is Lebesgue measurable, then $E_l - F$ is Lebesgue measurable, and so

$$\mu(E_l - F) = \mu_e(E_l - F) = 0.$$

The set F is a countable union of closed subsets $\{F_n\}$ of E_l.

Each F_n is compact because it is closed and belongs to the bounded E_l.

Since each F_n is a compact subset of $E_l \subset E$, and every compact subset of E has Lebesgue measure zero, we have that $\mu(F_n) = 0$ for all n.

By countable subadditivity we have

$$\mu(F) = \mu\left(\bigcup F_n\right) \leq \sum \mu(F_n) = 0.$$

Thus we have that

$$\mu(E_l) - \mu(F) = \mu(E_l - F) = 0.$$

But $\mu(F) = 0$ and $\mu(E_l) > 0$ making $\mu(E_l) - \mu(F) > 0$, a contradiction.

Thus E_l is not Lebesgue measurable.