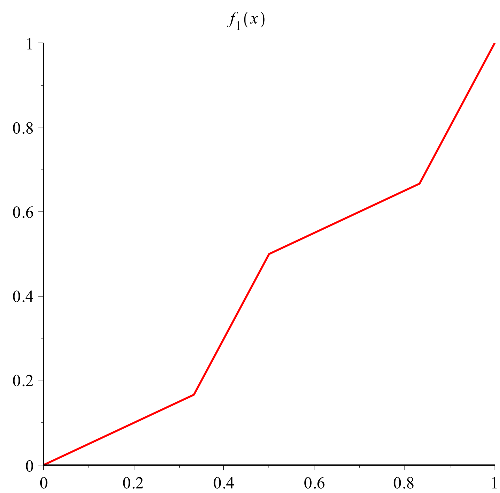
Math 541 Lecture #19
II.14: Borel Sets, Measurable Sets, and Incomplete Measures, Part I

We will answer in the affirmative, by way of an example, that there are Lebesgue measurable sets in $\mathbb{R}$ that are not Borel sets.

§14.1: A Continuous Increasing Function $f : [0,1] \to [0,1]$. We inductively construct a sequence of continuous increasing functions $f_n : [0,1] \to [0,1]$ starting with $f_0(x) = x$ and

$$f_1(x) = \begin{cases} \dfrac{x}{2} & \text{if } 0 \le x \le \dfrac{1}{3}, \\ 2x - \dfrac{1}{2} & \text{if } \dfrac{1}{3} \le x \le \dfrac{1}{2}, \\ \dfrac{x}{2} + \dfrac{1}{4} & \text{if } \dfrac{1}{2} \le x \le \dfrac{5}{6}, \\ 2x - 1 & \text{if } \dfrac{5}{6} \le x \le 1. \end{cases}$$

Here is the graph of $f_1(x)$.



The domain $[0,1]$ of f_1 is subdivided into 4^1 subintervals over which the graph of f is affine, i.e., a straight line segment, the slopes of which alternate between 2^{-1} and 2^1.

The graph of $f_1(x)$ lies below the graph of $f_0(x)$.

For each $n \ge 1$, the functions f_n satisfies the following.

(i) Each f_n is continuous and increasing on $[0,1]$, where the domain of f_n is subdivided into 4^n subintervals over which the graph of f is affine, the slopes of which alternate between 2^{-n} and 2^n.

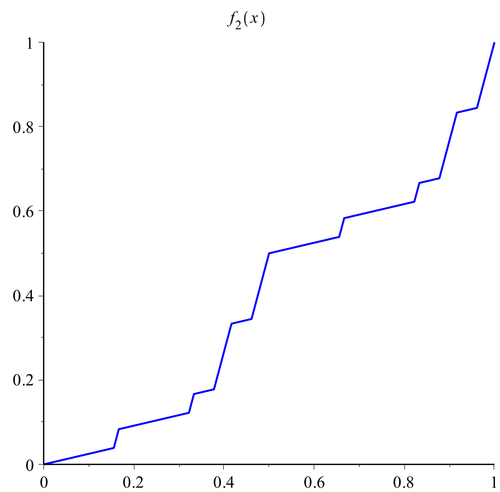
(ii) $f_{n+1}(x) \le f_n(x)$ for all $x \in [0,1]$.

(iii) If α is an endpoint of any of the 4^n subintervals into which $[0,1]$ has been subdivided, then $f_m(\alpha) = f_n(\alpha)$ for all $m \ge n$.

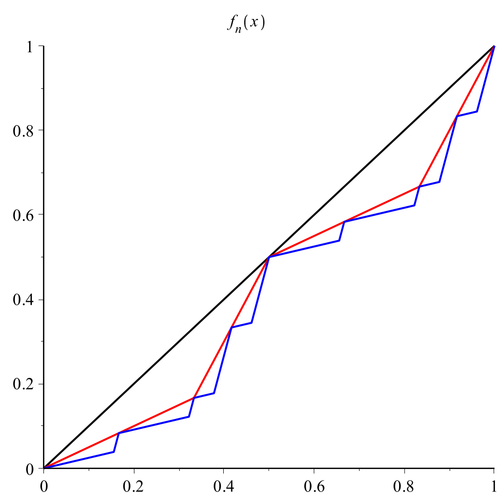
(iv) If $[\alpha, \beta]$ is an subinterval on which f_n is affine, then

$$\beta - \alpha \leq 2^{-n} \text{ and } f_n(\beta) - f_n(\alpha) \leq 2^{-n}.$$

Here is the graph of $f_2(x)$, where the slopes alternate between 2^{-2} and 2^2.



And here are the graphs of $f_0(x)$, $f_1(x)$, and $f_2(x)$ on the same set of axes.



We detail how f_{n+1} is induced by f_n for $n \geq 0$.

Let $[\alpha, \beta]$ be an interval on which f_n is affine, and set $A = (\alpha, f_n(\alpha))$ and $B = (\beta, f_n(\beta))$.

Let C be the midpoint of the line segment $\overline{AB}$.

Let D be the unique point below the line segment $\overline{AC}$ such that the slope of $\overline{AD}$ is 2^{-n-1} and the slope of $\overline{DC}$ is 2^{n+1}.

[Think of the line of slope 2^{-n-1} passing through A and the line of slope 2^{n+1} passing through C; they intersect in a unique point D below the line segment $\overline{AC}$.]

Similarly there is a unique point E that lies below the line segment $\overline{CB}$ such that the slope of the line segment $\overline{CE}$ is 2^{-n-1} and the slope of the line segment $\overline{EB}$ is 2^{n+1}.

The graph of f_{n+1} over $[\alpha, \beta]$ is the polygonal path $ADCEB$.

Since f_n is continuous and strictly increasing for each n, the inverse f_n^{-1} exists, is continuous and strictly increasing.

Moreover, the inverses $\{f_n^{-1}\}$ satisfy the following.

(i) Each f_n^{-1} is continuous and increasing on $[0, 1]$, where the domain of f_n^{-1} is subdivided into 4^n subintervals over which the graph of f is affine, the slopes of which alternate between 2^n and 2^{-n}.

(ii) $f_{n+1}^{-1}(x) \geq f_n^{-1}(x)$ for all $x \in [0, 1]$.

(iii) If α is an endpoint of any of the 4^n subintervals into which $[0, 1]$ has been subdivided, then $f_m^{-1}(\alpha) = f_n^{-1}(\alpha)$ for all $m \geq n$.

(iv) If $[\alpha, \beta]$ is an subinterval on which f_n^{-1} is affine, then

$$\beta - \alpha \leq 2^{-n} \text{ and } f_n^{-1}(\beta) - f_n^{-1}(\alpha) \leq 2^{-n}.$$

We have a sequence of continuous, strictly increasing, piecewise affine functions $f_n : [0, 1] \to [0, 1]$ whose inverses f_n^{-1} are continuous, strictly increasing, and piecewise affine.

Proposition. The sequence $\{f_n\}$ converges uniformly to a continuous, strictly increasing function $f : [0, 1] \to [0, 1]$.

Proof. For a fixed $n \geq 0$, each $x \in [0, 1]$ belongs to at least one of the 4^n subintervals of $[0, 1]$ on which f_n is affine.

Suppose $[\alpha, \beta]$ is one such subinterval to which x belongs.

By the monotonicity of f_n we have that $f_n(x) \leq f_n(\beta)$.

For $m \geq n$, we have by the monotonicity of f_m that $f_m(\alpha) \leq f_m(x)$, and so $-f_m(x) \leq -f_m(\alpha)$.

Since $m \geq n$, we have $f_m(\alpha) = f_n(\alpha)$, and so $-f_m(x) \leq -f_n(\alpha)$.

Then for all $m \geq n$ there holds

$$0 \leq f_n(x) - f_m(x) \leq f_n(\beta) - f_n(\alpha) \leq 2^{-n}.$$

This says that the sequence $\{f_n\}$ satisfies the Cauchy Criterion for uniform convergence: for every $\epsilon > 0$ the choice of $N \in \mathbb{N}$ such that $2^{-N} < \epsilon$, satisfies $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in [0, 1]$ and all $m, n \geq N$; and hence the sequence $\{f_n\}$ converges uniformly to a continuous function $f : [0, 1] \to [0, 1]$.

Since $f_n(x) \leq f_n(y)$ for all n and for all $x \leq y$, we have that the limit function f satisfies

$$f(x) = \lim_{n \to \infty} f_n(x) \leq \lim_{n \to \infty} f_n(y) = f(y).$$

This says that the limit function is nondecreasing on $[0, 1]$.

For $x < y$ in $[0, 1]$, there is a large enough n such that one of the 4^n subintervals $[\alpha, \beta]$ on which f_n is affine, is contained in $[x, y]$, i.e., $x \leq \alpha < \beta \leq y$.

Since $f_m(\alpha) = f_n(\alpha)$ for all $m \geq n$, we have that $f(\alpha) = f_n(\alpha)$; similarly we have that $f(\beta) = f_n(\beta)$.

Thus we obtain that

$$f(x) \leq f(\alpha) = f_n(\alpha) < f_n(\beta) = f(\beta) \leq f(y).$$

This says that the limit function f is strictly increasing on $[0, 1]$. $\square$

Corollary. The inverse f^{-1} of the limit function is continuous and strictly increasing on $[0, 1]$.