Math 541 Lecture #20
II.14: Borel Sets, Measurable Sets, and Incomplete Measures, Part II

§14.1A: Measure Theoretic Properties of f. We now explore some of the measure theoretic properties of the continuous, strictly increasing function f, with respect to Lebesgue measure μ on $\mathbb{R}$.

Set

$$A_n = \bigcup \{\text{intervals } [\alpha,\beta], \text{ where } f_n \text{ is affine and } f_n' = 2^n\},$$
$$B_n = \bigcup \{\text{intervals } [\alpha,\beta], \text{ where } f_n \text{ is affine and } f_n' = 2^{-n}\}$$

For an endpoint z of an interval $[\alpha,\beta] \in A_n$, we have $f(z) = f_n(z)$; hence

$$[\alpha,\beta] \in A_n \Rightarrow f(\beta) - f(\alpha) = f_n(\beta) - f_n(\alpha) = 2^n(\beta-\alpha),$$
$$[\alpha,\beta] \in B_n \Rightarrow f(\beta) - f(\alpha) = f_n(\beta) - f_n(\alpha) = 2^{-n}(\beta-\alpha).$$

Since Lebesgue measure of an interval is the length of the interval, we have

$$\mu(f([\alpha,\beta])) = f(\beta) - f(\alpha) = 2^n(\beta-\alpha) = 2^n\mu([\alpha,\beta]),$$
$$\mu(f([\alpha,\beta])) = f(\beta) - f(\alpha) = 2^{-n}(\beta-\alpha) = 2^{-n}\mu([\alpha,\beta]).$$

Since the finitely many intervals in A_n are pairwise disjoint and f is strictly increasing, the images of those intervals by f are also pairwise disjoint.

Similarly, the finitely many intervals in B_n are pairwise disjoint, and so the strict monotonicity of f implies the images of those intervals in B_n are also pairwise disjoint.

Thus for $A_n = [\alpha_1,\beta_1] \cup \cdots \cup [\alpha_k,\beta_k]$, we have

$$f_n(A_n) = [f(\alpha_1), f(\beta)] \cup \cdots \cup [f(\alpha_k), f(\beta_k)];$$

a similar statement holds for B_n and $f_n(B_n)$.

Finite additivity (implied by countable additivity of μ) gives

$$\mu(f(A_n)) = 2^n\mu(A_n),$$
$$\mu(f(B_n)) = 2^{-n}\mu(B_n).$$

Since $[0,1] = A_n \cup B_n$ and since $A_n \cap B_n$ is a finite set (the common endpoints of the intervals) and hence of Lebesgue measure 0, we have

$$1 = \mu([0,1]) = \mu(A_n \cup B_n) = \mu(A_n) + \mu(B_n) - \mu(A_n \cap B_n) = \mu(A_n) + \mu(B_n).$$

Similarly, since $[0,1] = f(A_n) \cup f(B_n)$, and since $f(A_n) \cap f(B_n)$ is a finite set, we have

$$1 = \mu(f(A_n)) + \mu(f(B_n)).$$

Since $\mu(f(A_n)) = 2^n\mu(A_n)$ and $\mu(f(B_n)) = 2^{-n}\mu(B_n)$, we obtain

$$1 = 2^n\mu(A_n) + 2^{-n}\mu(B_n).$$

Thus we have two linear equations in the two unknowns $\mu(A_n)$ and $\mu(B_n)$:

$$\begin{array}{rcrcl} \mu(A_n) & + & \mu(B_n) & = & 1 \\ 2^n\mu(A^n) & + & 2^{-n}\mu(B_n) & = & 1. \end{array}$$

Using Cramer's Rule we obtain

$$\mu(A_n) = \frac{2^{-n}-1}{2^{-n}-2^n} = \frac{2^{-n}-1}{2^{-n}-2^n}\left(\frac{-2^n}{-2^n}\right) = \frac{2^n-1}{2^{2n}-1},$$
$$\mu(B_n) = \frac{1-2^n}{2^{-n}-2^n} = \frac{1-2^n}{2^{-n}-2^n}\left(\frac{-2^n}{-2^n}\right) = 2^n\frac{2^n-1}{2^{2n}-1}.$$

Since $\mu(f(A_n)) = 2^n\mu(A_n)$ and $\mu(f(B_n)) = 2^{-n}\mu(B_n)$, we also obtain

$$\mu(f(A_n)) = 2^n\frac{2^n-1}{2^{2n}-1} = \mu(B_n),$$
$$\mu(f(B_n)) = \frac{2^n-1}{2^{2n}-1} = \mu(A_n).$$

Set

$$\mathcal{S}_n = \bigcup_{j=n}^{\infty} A_j, \quad \mathcal{S} = \bigcap_{n=1}^{\infty} \mathcal{S}_n = \bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty} A_n = \limsup A_n.$$

Since each A_n is a union of intervals, it is measurable; hence for each n the set $\mathcal{S}_n$ is measurable; finally $\mathcal{S}$ is measurable.

Since $\emptyset \subset \mathcal{S} \subset \mathcal{S}_n$ for all n, and since $\mathcal{S}_n = \cup_{j=n}^{\infty} A_j$ (not disjointly), we have

$$0 \le \mu(\mathcal{S}) \le \mu(\mathcal{S}_n) \le \sum_{j=n}^{\infty} \mu(A_j).$$

The tail of the series goes to 0 as $n \to \infty$ because

$$\sum_{j=n}^{\infty} \mu(A_j) = \sum_{j=n}^{\infty} \frac{2^j-1}{2^{2j}-1} = \sum_{j=n}^{\infty} \frac{2^j-1}{(2^j-1)(2^j+1)} = \sum_{j=n}^{\infty} \frac{1}{2^j+1} \le \sum_{j=n}^{\infty} \frac{1}{2^j} = \frac{1}{2^{n-1}} \to 0.$$

By the Squeeze Theorem we have $\mu(\mathcal{S}) = 0$.

Now we show that $f(\mathcal{S})$ is measurable and determine the value of $\mu(f(\mathcal{S}))$.

Each set $f(A_n)$ is a finite union of closed intervals because f is continuous and strictly increasing.

Thus $f(A_n)$ is measurable, making the sets

$$f(\mathcal{S}_n) = \bigcup_{j=n}^{\infty} f(A_j), \; f(\mathcal{S}) = \bigcap_{n=1}^{\infty} f(\mathcal{S}_n) = \limsup f(A_n)$$

measurable as well.

Since $\cup_{n=1}^{\infty} f(A_n) \subset [0,1]$, we have $\mu(\cup f(A_n)) < \infty$.

Thus by Proposition 3.1 we have

$$\mu(f(\mathcal{S})) = \mu(\limsup f(A_n)) \geq \limsup \mu(f(A_n)) = \limsup 2^n \frac{2^n - 1}{2^{2n} - 1}.$$

The sequence in the limsup is a convergent sequence with limit 1 becasue

$$2^n \frac{2^n - 1}{2^{2n} - 1} = \frac{2^{2n} - 2^n}{2^{2n} - 1} = \frac{2^{2n} - 2^n}{2^{2n} - 1} \left(\frac{2^{-2n}}{2^{-2n}} \right) = \frac{1 - 2^{-n}}{1 - 2^{-2n}} \to 1.$$

We obtain that $1 \leq \mu(f(\mathcal{S}))$.

On the other hand, since $f(\mathcal{S}) \subset [0,1]$, we have $\mu(f(\mathcal{S})) \leq 1$.

By the Squeeze Theorem, we obtain $\mu(f(\mathcal{S})) = 1$.

Therefore, the function f maps the set $\mathcal{S}$ of measure 0 to the set $f(\mathcal{S})$ of measure 1.

Likewise, the function f maps the set $[0,1] - \mathcal{S}$ of measure 1 to the set $[0,1] - f(\mathcal{S})$ of measure 0 because (using the injectivity of f) we have

$$\begin{aligned} f([0,1] - \mathcal{S}) &= f([0,1] \cap \mathcal{S}^c) = f([0,1]) \cap f(\mathcal{S}^c) \\ &= [0,1] \cap [f(\mathcal{S})]^c = [0,1] - f(\mathcal{S}), \end{aligned}$$

so that

$$\mu(f([0,1] - \mathcal{S})) = \mu([0,1] - f(\mathcal{S})) = 1 - \mu(f(\mathcal{S})) = 0.$$