Math 541 Lecture #21
II.14: Borel Sets, Measurable Sets, and Incomplete Measures, Part III

14.2: On the Preimage of a Measurable Set. Recall that a subset E of $[0,1]$ is relatively open if there is an open subset O of $\mathbb{R}$ such that $E = O \cap [0,1]$.

For $f : [0,1] \to [0,1]$ continuous, the preimage $f^{-1}(O)$ of a relatively open set O in $[0,1]$ is a relatively open set in $[0,1]$, and hence the preimage is Lebesgue measurable.

Similarly the preimage of a relatively closed set is relatively closed, and hence the preimage of a relatively closed set is Lebesgue measurable.

More generally we consider the collection $\mathcal{F}$ of subsets E of $[0,1]$ for which the preimage $f^{-1}(E)$ (a subset of $[0,1]$) is Lebesgue measurable.

Proposition. The collection $\mathcal{F}$ is a σ-algebra of subsets of $[0,1]$ that contains Borel subsets of $[0,1]$.

Proof. We are to show that the relative complement of any element of $\mathcal{F}$ (relative to $[0,1]$) is in $\mathcal{F}$, and that the union of a countable collection of elements in $\mathcal{F}$ is in $\mathcal{F}$.

For $E \in \mathcal{F}$ we have that $f^{-1}(E)$ is Lebesgue measurable.

For the relative complement $[0,1]-E$ to belong to $\mathcal{F}$, we are to show that $f^{-1}([0,1]-E)$ is Lebesgue measurable.

Using properties of preimages of functions on intersections and complements we have

$$\begin{aligned} f^{-1}([0,1]-E) &= f^{-1}([0,1] \cap E^c) \\ &= f^{-1}([0,1]) \cap f^{-1}(E^c) \\ &= [0,1] \cap (f^{-1}(E))^c \\ &= [0,1] - f^{-1}(E). \end{aligned}$$

Since $f^{-1}(E)$ is Lebesgue measurable, so then is $[0,1] - f^{-1}(E) = f^{-1}([0,1]-E)$, and hence $[0,1]-E \in \mathcal{F}$.

Now take a countable collection $\{E_n\}$ of elements in $\mathcal{F}$.

Then for each n we have $f^{-1}(E_n)$ is Lebesgue measurable, so that

$$f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(E_n)$$

is Lebesgue measurable, being the countable union of Lebesgue measurable sets.

Thus $\mathcal{F}$ is σ-algebra.

Each relatively open subset of $[0,1]$ belongs to $\mathcal{F}$ because its preimage is relatively open.

Since the σ-algebra $\mathcal{B}$ of Borel subsets of $[0,1]$ is the smallest σ-algebra containing the relatively open subsets of $[0,1]$, it follows that $\mathcal{B} \subset \mathcal{F}$. $\square$

Proposition. If E is a Borel subset of $[0,1]$, then $f^{-1}(E)$ is a Borel subset of $[0,1]$.

Proof. We will show that $\Omega = \{E \subset [0,1] : f^{-1}(E) \in \mathcal{B}\}$ contains $\mathcal{B}$.

By an argument similar to that used in the proof of the previous Proposition, we show that Ω is a σ-algebra in $[0,1]$.

Each relatively open set $\mathcal{O}$ in $[0,1]$ belongs to Ω because, by the continuity of f, we have $f^{-1}(O)$ is relatively open and hence in $\mathcal{B}$.

Since Ω is a σ-algebra containing all of the relatively open subsets of $[0,1]$, we have $\mathcal{B} \subset \Omega$.

Thus, each every Borel subset E of $[0,1]$ belongs to Ω, and it has the property that $f^{-1}(E) \in \mathcal{B}$. □

14.3: Proofs of Two Propositions. We have now developed enough to prove, using the continuous, strictly increasing function $f : [0,1] \to [0,1]$, the existence of a Lebesgue measurable subset of $\mathbb{R}$ that is not a Borel set, and the existence of a Borel measure that is not complete.

Proposition 14.1. There exists a Lebesgue measurable subset $\mathcal{D}$ of $[0,1]$ which is not a Borel set, and whose preimage under f is not Lebesgue measurable.

Proof. Recall that there is a Lebesgue measurable subset $\mathcal{S}$ of $[0,1]$ with Lebesgue measure 0 whose image $f(\mathcal{S})$ is Lebesgue measurable with Lebesgue measure 1.

Furthermore, the function f maps the Lebesgue measurable set $[0,1] - \mathcal{S}$ of Lebesgue measure 1 to the Lebesgue measurable set $[0,1] - f(\mathcal{S})$ of Lebesgue measure zero.

Since Lebesgue measure is complete, every subset of $\mathcal{S}$ is Lebesgue measurable and has Lebesgue measure zero.

Likewise, every subset of $[0,1] - f(\mathcal{S})$ is Lebesgue measurable and has Lebesgue measure zero.

Let E be the Vitali subset of $[0,1]$ that is not Lebesgue measurable.

The set $E \cap \mathcal{S}$ is Lebesgue measurable because Lebesgue measure is complete: the set $E \cap \mathcal{S}$ is a subset of a Lebesgue measurable set of meassure zero.

The set $E - \mathcal{S}$ is not Lebesgue measurable, because if it were, then E would be the (disjoint) union of the Lebesgue measurable sets $E - \mathcal{S}$ and $E \cap \mathcal{S}$.

The set $\mathcal{D} = f(E - \mathcal{S})$ is contained in $[0,1] - f(\mathcal{S})$ because $f(E) \subset [0,1]$ and by the injectivity of f we have

$$f(E - \mathcal{S}) = f(E \cap \mathcal{S}^c) = f(E) \cap f(\mathcal{S}^c) = f(E) \cap [f(\mathcal{S})]^c = f(E) - f(\mathcal{S}).$$

Since $[0,1] - f(\mathcal{S})$ is a set of Lebesgue measure zero, $f(E) - f(\mathcal{S}) \subset [0,1] - f(\mathcal{S})$, and Lebesgue measure is complete, the set $\mathcal{D}$ is Lebesgue measurable with Lebesgue measure zero.

The preimage of $\mathcal{D}$ is not Lebesgue measurable because f is invertible so that

$$f^{-1}(\mathcal{D}) = f^{-1}\big(f(E - \mathcal{S})\big) = E - \mathcal{S},$$

which is not measurable.

If the Lebesgue measurable set $\mathcal{D}$ were a Borel set, then by the previous Proposition, the preimage $f^{-1}(\mathcal{D})$ would be a Borel set, and hence Lebesgue measurable.

By this contradiction, the set $\mathcal{D}$ is not a Borel set. $\square$

Proposition 14.2. The restriction of Lebesgue measure on $\mathbb{R}$ to the σ-algebra of Borel sets in $\mathbb{R}$ is not a complete measure.

Proof. Let $\mathcal{D}$ be the Lebesgue measurable set of Lebesgue measure zero, as given in the proof of Proposition 14.1.

By Proposition 12.3, there is a set $\mathcal{D}_\delta$ of type $\mathcal{G}_\delta$ such that $\mathcal{D} \subset \mathcal{D}_\delta$ and

$$\mu(\mathcal{D}_\delta) = \mu(\mathcal{D}_\delta) - \mu(\mathcal{D}) = \mu(\mathcal{D}_\delta - \mathcal{D}) = 0.$$

The set $\mathcal{D}_\delta$ is a Borel set that has Lebesgue measure zero, but it contains the subset $\mathcal{D}$ that is not a Borel set.

Thus the restriction of Lebesgue measure μ to the σ-algebra of Borel sets in $\mathbb{R}$ is not a complete measure. $\square$

Recall that $\mathcal{F}$ is the σ-algebra of subsets E of $[0,1]$ whose preimages $f^{-1}(E)$ are Lebesgue measurable.

By way of a slight abuse of notation, let $\mathcal{M}$ denote the σ-algebra of Lebesgue measurable subsets of $[0,1]$.

Then

$$\mathcal{F} = \{E \subset [0,1] : f^{-1}(E) \in \mathcal{M}\}.$$

Proposition. For the function f, there holds $\mathcal{M} \not\subset \mathcal{F}$.

Proof. By Proposition 14.1, we have the existence of $\mathcal{D} \in \mathcal{M}$ for which $f^{-1}(\mathcal{D}) \notin \mathcal{M}$.

Thus $\mathcal{D} \notin \mathcal{F}$, which implies that $\mathcal{M} \not\subset \mathcal{F}$. $\square$