Math 541 Lecture #22
II.15: More on Borel Measures, Part I

Recall that a Borel measure μ on X is a measure whose σ-algebra domain contains the Borel sets of X.

[Note: some authors in other books define a Borel measure to be a measure whose domain is precisely the Borel sets of X.]

The Lebesgue measure on $\mathbb{R}^N$ is a Borel measure which has the properties that Lebesgue measurable sets E of finite Lebesgue measure can be approximated by the Lebesgue measure of open sets containing E or closed subsets of E. [This is the content of Propositions 12.3 and 12.4.]

To what extend can this approximation be done for an arbitrary Borel measure?

Example. The non σ-finite counting measure on $\mathbb{R}$ is a Borel measure because it is defined on σ-algebra of all subsets of $\mathbb{R}$, hence on the Borel sets.

A singleton set has a counting measure value of 1, but every open set, being a infinite subset, has counting measure value of ∞.

Thus the counting measure values of opens sets do not approximate the counting measure value of all Borel sets.

But there are some Borel measures for which the approximation can be done.

Proposition 15.1. Let μ be a finite Borel measure in $\mathbb{R}^N$, i.e., $\mu(\mathbb{R}^N) < \infty$, and let E be a Borel set.

(1) For all $\epsilon > 0$ there exists a closed subset $E_{c,\epsilon}$ such that $E_{c,\epsilon} \subset E$ and $\mu(E - E_{c,\epsilon}) \leq \epsilon$.

(2) For all $\epsilon > 0$, there exists a open subset $E_{o,\epsilon}$ such that $E \subset E_{o,\epsilon}$ and $\mu(E_{0,\epsilon} - E) \leq \epsilon$.

Proof. (1) Let $\mathcal{A}$ be the σ-algebra domain of the finite Borel measure μ in $\mathbb{R}^N$.

Let $\mathcal{C}_0$ be the collection of sets $E \in \mathcal{A}$ such that for every $\epsilon > 0$ there exists a closed set $C \subset E$ such that $\mu(E - C) \leq \epsilon$.

This collection is nonempty because for each closed set C we have $C \subset C$ and $\mu(C - C) = 0 \leq \epsilon$.

We will show that $\mathcal{C}_0$ contains a σ-algebra that contains all of the open sets, and hence all Borel sets.

<u>Claim 1</u>. Countable intersections of elements of $\mathcal{C}_0$ are in $\mathcal{C}_0$.

Let $\{E_n\}$ be a countable collection of sets in $\mathcal{C}_0$.

Having fixed $\epsilon > 0$, select closed sets $C_n \subset E_n$ such that $\mu(E_n - C_n) \leq 2^{-n}\epsilon$.

The inclusion

$$\bigcap_{n=1}^{\infty} E_n - \bigcap_{n=1}^{\infty} C_n \subset \bigcup_{n=1}^{\infty} (E_n - C_n)$$

holds because for $x \in \cap E_n - \cap C_n$ we have $x \in \cap E_n$ but $x \notin \cap C_n$, meaning $x \in E_n$ for all n, while $x \in \cup C_n^c$, meaning there is some m for which $x \in C_m^c$; for this m we then have $x \in E_m$ and $x \in C_m^c$, meaning that $x \in E_m$ but $x \notin C_m$, and hence $x \in \cup(E_n - C_n)$.

By the monotonicity and subadditivity of μ we have

$$\mu\left(\bigcap_{n=1}^{\infty} E_n - \bigcap_{n=1}^{\infty} C_n\right) \leq \mu\left(\bigcup_{n=1}^{\infty}(E_n - C_n)\right) \leq \sum_{n=1}^{\infty} \mu(E_n - C_n) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Since $\cap C_n$ is closed, the set $\cap E_n$ belongs to $\mathcal{C}_0$, and we have Claim 1.

Claim 2. Countable unions of elements of $\mathcal{C}_0$ are in $\mathcal{C}_0$.

Let $\{E_n\}$ be a countable collection in $\mathcal{C}_0$, and for a fixed $\epsilon > 0$ select closed sets $C_n \subset E_n$ for which $\mu(E_n - C_n) \leq \epsilon/2^n$.

The sequence of sets

$$A_m = \bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{m} C_n$$

is monotone decreasing, and $\mu(A_1) < \infty$ because μ is a finite measure, and so

$$\lim_{m\to\infty} \mu(A_m) = \mu\left(\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{\infty} C_n\right).$$

The inclusion

$$\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{\infty} C_n \subset \bigcup_{n=1}^{\infty}(E_n - C_n)$$

holds because for x belonging to the left-hand side we have $x \in E_m$ for some m while $x \notin \cup C_n$, whence $x \in E_m$ and $x \in \cap C_n^c$, so that $x \in E_m$ and $x \in C_m^c$, thus giving $x \in E_m$ and $x \notin C_m$, i.e., $x \in E_m - C_m$.

By monotonicity and countably subadditivity of the measure we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{\infty} C_n\right) \leq \mu\left(\bigcup_{n=1}^{\infty}(E_n - C_n)\right) \leq \sum_{n=1}^{\infty} \mu(E_n - C_n) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Thus $\lim \mu(A_m)$ converges to a value no bigger than ϵ, and so there is $m \in \mathbb{N}$ such that $\mu(A_m) \leq 2\epsilon$.

The union $\cup_{n=1}^m C_n$ is a closed subset because it is a finite union of closed subsets.

Thus the union $\cup E_n$ belongs to $\mathcal{C}_0$, and we have Claim 2.

The collection $\mathcal{C}_0$ trivially contains all of the closed sets, and in particular, the closed dyadic cubes.

Every open set is a countable union of closed dyadic cubes.

Since by Claim 2, the countable union of elements of $\mathcal{C}_0$ is in $\mathcal{C}_0$, we have that $\mathcal{C}_0$ contains all of the open sets.

How do we get "closed under complements" for $\mathcal{C}_0$?

Set

$$\mathcal{C} = \{E \in \mathcal{C}_0 : E^c \in \mathcal{C}_0\}.$$

This collection $\mathcal{C}$ is closed under taking complements: if $E \in \mathcal{C}$, then $E \in \mathcal{C}_0$ with $E^c \in \mathcal{C}_0$, so that $E^c \in \mathcal{C}$ because $E^c \in \mathcal{C}_0$ and $(E^c)^c = E_c \in \mathcal{C}_0$.

In particular, since the complement of a closed set is open, and every open set is in $\mathcal{C}_0$, we have that every closed set is in $\mathcal{C}$.

Similarly every open set is in $\mathcal{C}$.

For $\mathcal{C}$ to be a σ-algebra, it remains to show that $\mathcal{C}$ is closed under countable unions.

Let $\{E_n\}$ be a countable collection in $\mathcal{C}$.

Then for each n we have E_n and E_n^c both belong to $\mathcal{C}_0$.

By Claim 2, we have $\cup E_n \in \mathcal{C}_0$, and by Claim 1 we have

$$\left(\bigcup_{n=1}^{\infty} E_n\right)^c = \bigcap_{n=1}^{\infty} E_n^c \in \mathcal{C}_0.$$

Thus $\cup E_n$ belongs to $\mathcal{C}$.

Therefore $\mathcal{C}$ is a σ-algebra that contains the open sets, and hence the Borel sets.

(2) Let E be a Borel set.

Then E^c is a Borel set, and there is for every $\epsilon > 0$ by (1) a closed set C such that $C \subset E^c$ and $\mu(E^c - C) \leq \epsilon$.

Since

$$\begin{aligned}
E^c - C &= (\mathbb{R}^N - E) - C \\
&= (\mathbb{R}^N \cap E^c) - C \\
&= (\mathbb{R}^N \cap E^c) \cap C^c \\
&= (\mathbb{R}^N \cap C^c) \cap E^c \\
&= (\mathbb{R}^N - C) \cap E^c \\
&= (\mathbb{R}^N - C) - E \\
&= C^c - E,
\end{aligned}$$

we have

$$\mu(C^c - E) = \mu(E^c - C) \leq \epsilon.$$

The open set C^c satisfies $E \subset C^c$ and $\mu(C^c - E) \leq \epsilon$. □

Corollary 15.2. Let μ be a finite Borel measure in $\mathbb{R}^N$ and let E be a Borel set. Then there exists a

(1) a set E_σ of type $\mathcal{F}_\sigma$ such that $E_\sigma \subset E$ and $\mu(E - E_\sigma) = 0$, and
(2) a set E_δ of type $\mathcal{G}_\delta$ such that $E \subset E_\delta$ and $\mu(E_\delta - E) = 0$.

The proof of this Corollary is similar to what we saw before in the approximation theory of measurable sets.