Math 541 Lecture #23
II.15: More on Borel Measures, Part II

§15.1: Some Extensions to general Borel Measures. The approximation with closed sets contained in a Borel set E continues to hold for Borel measures that are not necessarily finite, provided $\mu(E) < \infty$.

Proposition 15.3. Let μ be a Borel measure in $\mathbb{R}^N$ and let E be a Borel set of finite measure. For every $\epsilon > 0$ there exists a closed set $E_{c,\epsilon}$ such that $E_{c,\epsilon} \subset E$ and $\mu(E - E_{c,\epsilon}) \leq \epsilon$.

Proof. Let $\mathcal{A}$ be the σ-algebra on which μ is defined.

For a fixed Borel set E of finite measure, define

$$\mu_E(A) = \mu(E \cap A), \ \ A \in \mathcal{A}.$$

This set function on $\mathcal{A}$ is a measure because $\mu_E(A) = \mu(E \cap A) \geq 0$ for all $A \in \mathcal{A}$, for a countable collection of pairwise disjoint $\{A_n\}$ in $\mathcal{A}$ there holds

$$\begin{aligned}\mu_E(\cup A_n) &= \mu\left(E \cap \left(\bigcup A_n\right)\right) \\ &= \mu\left(\bigcup (E \cap A_n)\right) \\ &= \sum \mu(E \cap A_n) \\ &= \sum \mu_E(A_n),\end{aligned}$$

and $\mu_E(\emptyset) = \mu(E \cap \emptyset) = 0$ is finite.

The measure μ_E is a finite Borel measure because $\mu_E(X) = \mu(E) < \infty$ and the domain of μ_E is $\mathcal{A}$ which contains the Borel sets.

Thus by Proposition 15.1 part (1) for each $\epsilon > 0$ there is a closed set $E_{c,\epsilon}$ such that $E_{c,\epsilon} \subset E$ and $\mu_E(E - E_{c,\epsilon}) \leq \epsilon$.

Since $E \cap (E - E_{c,\epsilon}) = E - E_{c,\epsilon}$, we have by the definition of μ_E that

$$\mu(E - E_{c,\epsilon}) = \mu(E \cap (E - E_{c,\epsilon})) = \mu_E(E - E_{c,\epsilon}) \leq \epsilon,$$

giving the result. □

Unfortunately, extending the existence of an open set $E_{o,\epsilon}$ such that $E \subset E_{o,\epsilon}$ and $\mu(E_{o,\epsilon} - E) \leq \epsilon$ to Borel sets E of finite measure fails, as the counting measure example illustrates.

However, we can get this approximation to hold certain Borel measure that are not finite.

Proposition 15.4. Let μ be a Borel measure in $\mathbb{R}^N$ that is finite on bounded sets, and let E be a Borel set of finite measure. For every $\epsilon > 0$ there exists an open set $E_{o,\epsilon}$ such that $E \subset E_{o,\epsilon}$ and $\mu(E_{o,\epsilon} - E) \leq \epsilon$.

Proof. Suppose E is a Borel set of finite measure.

For each $n \in \mathbb{N}$, let Q_n be the open cube centered at the origin, with edge having length n, and with faces parallel to the coordinate planes.

Since E and Q_n are Borel sets, the difference $Q_n - E$ is a bounded Borel set, for which by hypothesis we have $\mu(Q_n - E) < \infty$.

For $\epsilon > 0$ there is by Proposition 15.3 a closed set $C_n \subset (Q_n - E)$ such that

$$\mu((Q_n - E) - C_n) \leq \frac{\epsilon}{2^n}.$$

Since

$$(Q_n - E) - C_n = (Q_n \cap E^c) \cap C_n^c = (Q_n \cap C_n^c) \cap E^c = (Q_n - C_n) - E,$$

we have

$$\mu((Q_n - C_n) - E) = \mu((Q_n - E) - C_n) \leq \frac{\epsilon}{2^n}.$$

The set $Q_n - C_n$ is open because Q_n is open and C_n is closed.

Claim. The set $Q_n - C_n$ contains $Q_n \cap E$.

For $x \in Q_n \cap E$ we have $x \in Q_n$ and $x \in E$.

Since $C_n \subset (Q_n - E)$ we have $(Q_n - E)^c \subset C_n^c$, since $(Q_n - E)^c = (Q_n \cap E^c)^c = Q_n^c \cup E$, and since $x \in E$, we have $x \in Q_n^c \cup E \subset C_n^c$.

Hence $x \notin C_n$, so that as $x \in Q_n$ we have $x \in Q_n - C_n$, giving the Claim.

Because $\mathbb{R}^N = \cup Q_n$, we have by the Claim that

$$E = \bigcup (Q_n \cap E) \subset \bigcup (Q_n - C_n).$$

Since each $Q_n - C_n$ is open the union $E_{o,\epsilon} = \cup(Q_n - C_n)$ is open and contains E.

Therefore,

$$\mu(E_{o,\epsilon} - E) = \mu\left(\bigcup (Q_n - C_n) - E\right) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

This gives for each $\epsilon > 0$ the existence of an open set $E_{o,\epsilon}$ such that $E \subset E_{o,\epsilon}$ and $\mu(E_{o,\epsilon} - E) \leq \epsilon$. □

§15.2. Regular Borel Measures and Radon Measures.

As we have seen, not every Borel measure μ has the property that $\mu(E) = \inf\{\mu(O) : E \subset O, O \text{ is open}\}$, e.g., the counting measure on $\mathbb{R}$.

Definition. A Borel measure μ on $\mathbb{R}^N$ is regular if for every Borel set E there holds

$$\mu(E) = \inf\{\mu(O) : E \subset O, O \text{ is open}\}.$$

In other sources this regularity of a Borel measure is called "outer regularity."

The Lebesgue measure in $\mathbb{R}^N$ is regular by Proposition 12.2.

Definition. A Borel measure on $\mathbb{R}^N$ is called a Radon measure if it is finite on compact subsets.

The Lebesgue measure in $\mathbb{R}^N$ is a Radon measure, as is the Lebesgue-Stieltjes measure in $\mathbb{R}$, and the Dirac measure.

The counting measure on $\mathbb{R}^N$ is Borel measure in $\mathbb{R}^N$ that is not a Radon measure since it is infinite on any non-finite compact subset.

The Hausdorff measure μ_α is a Borel measure in $\mathbb{R}^N$ that is not a Radon measure for all $\alpha \in [0, N)$.

Corollary. A Radon measure on $\mathbb{R}^N$ is a regular Borel measure.

Proof. Let E be a Borel set.

If $\mu(E) = \infty$, then by monotonicity, we have $\mu(O) = \infty$ for every open set O containing E.

Since $\mathbb{R}^N$ is open and contains E, we have

$$\mu(E) = \inf\{\mu(O) : E \subset O, O \text{ is open}\}.$$

Now suppose that $\mu(E) < \infty$.

To apply Proposition 15.4 to E, we need to show that μ is finite on bounded sets.

Since μ is a Radon measure, it is finite on compact sets.

But since the closure of any bounded set is compact, we have that μ is finite on bounded sets.

Now we can apply Proposition 15.4: for each $\epsilon > 0$ there is an open set $E_{o,\epsilon}$ such that $E \subset E_{o,\epsilon}$ and $\mu(E_{o,\epsilon} - E) \leq \epsilon$.

Because $\mu(E) < \infty$ we have

$$\mu(E_{o,\epsilon}) - \mu(E) = \mu(E_{o,\epsilon} - E) \leq \epsilon.$$

Thus we have for each $\epsilon > 0$ the existence of an open set $E_{o,\epsilon}$ such that $E \subset E_{o,\epsilon}$ and

$$\mu(E_{o,\epsilon}) \leq \mu(E) + \epsilon.$$

Since for any open set O containing E we have $\mu(E) \leq \mu(O)$ by monotonicity, the number $\mu(E)$ is a lower bound for the set $\{\mu(O) : E \subset O, O \text{ open}\}$.

These implies that $\mu(E) = \inf\{\mu(O) : E \subset O, O \text{ is open}\}$.

Therefore a Radon measure is regular. □

Homework Problem 23A. A Borel measure μ is called **inner regular** on a Borel set E if

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ is compact}\}.$$

Prove that if a Radon measure μ on $\mathbb{R}^N$ is inner regular on all open sets, then μ is inner regular on all Borel sets.

Homework Problem 23B. Regularity of a Borel measure μ is determined by its inner and outer regular on Borel sets. The definition of inner regularity on a Borel set is given in Problem 23A. The definition of **outer regularity** of μ on a Borel set E is that

$$\mu(E) = \inf\{\mu(O) : O \supset E, O \text{ open}\}$$

holds. A Borel measure is called "**regular**" if it is outer regular on all Borel sets and inner regularity on all Borel sets. (This is not the definition of regular in the text.) We proved that Lebesgue measure μ on $\mathbb{R}^N$ is outer regular on every Borel set. If Lebesgue measure is inner regular on open sets, then by 23A it would be inner regular on all Borel sets, making Lebesgue measure a "regular" Borel measure. Prove that Lebesgue measure is inner regular on open sets.