# Math 541 Lecture #24
## III.1: Measurable Functions, Part I

Let $\mathcal{A}$ be a $\sigma$-algebra on a set $X$.

For $E \in \mathcal{A}$, a function $f : E \to \mathbb{R}^*$, and $c \in \mathbb{R}$, define the sets

$$\begin{aligned}
[f > c] &= \{x \in E : f(x) > c\} = \{x \in E : f(x) \in (c, \infty]\} = f^{-1}((c, \infty]),\\
[f \geq c] &= \{x \in E : f(x) \geq c\} = \{x \in E : f(x) \in [c, \infty]\} = f^{-1}([c, \infty]),\\
[f < c] &= \{x \in E : f(x) < c\} = \{x \in E : f(x) \in [-\infty, c)\} = f^{-1}([\infty, c)),\\
[f \leq c] &= \{x \in E : f(x) \leq c\} = \{x \in E : f(x) \in [-\infty, c]\} = f^{-1}([\infty, c]).
\end{aligned}$$

We recognize that the first of these $[f > c]$ can be used to define directly or indirectly the other three:

$$\begin{aligned}
[f \geq c] &= \bigcap_{n=1}^{\infty} \left[ f > c - \frac{1}{n} \right],\\
[f \leq c] &= E - [f > c],\\
[f < c] &= E - [f \geq c].
\end{aligned}$$

Thus, if $[f > c]$ is measurable, i.e., in $\mathcal{A}$, for all $c \in \mathbb{R}$, then the sets $[f \geq c]$, $[f \leq c]$, and $[f < c]$ are also measurable for all $c \in \mathbb{R}$.

Now suppose that $[f \geq c]$ is measurable for all $c \in \mathbb{R}$.

Then since

$$[f > c] = \bigcup_{n=1}^{\infty} \left[ f \geq c + \frac{1}{n} \right],$$

it follows that $[f > c]$ is measurable for all $c \in \mathbb{R}$.

Similarly, if any one of the four sets $[f > c]$, $[f \geq c]$, $[f < c]$, or $[f \leq c]$ is measurable for all $c \in \mathbb{R}$, then all of the remaining three are also measurable for all $c \in \mathbb{R}$.

Definition. For a $\sigma$-algebra $\mathcal{A}$ in $X$, and $E \in \mathcal{A}$, a function $f : E \to \mathbb{R}^*$ is measurable if at least one of $[f > c]$, $[f \geq c]$, $[f < c]$, or $[f \leq c]$ is measurable for all $c \in \mathbb{R}$.

Note that the notion of the measurability of a function is independent of the measure $\mu$ we choose on $\mathcal{A}$.

Proposition 1.1. A function $f : E \to \mathbb{R}^*$ is measurable if and only if at least one of $[f > c]$, $[f \geq c]$, $[f < c]$, or $[f \leq c]$ is measurable for all $c \in \mathbb{Q}$.

Idea of Proof. Suppose that $[f \geq c]$ is measurable for all $c \in \mathbb{Q}$.

Fixing some $c \in \mathbb{R} - \mathbb{Q}$, let $\{q_n\} \subset \mathbb{Q}$ be a sequence converging in a monotonic decreasing manner to $c$.

Then

$$[f > c] = \bigcup_{n=1}^{\infty} [f \geq q_n]$$

is measurable. $\square$

Proposition 1.2. Let $f : E \to \mathbb{R}^*$ be measurable, and let $\alpha \in \mathbb{R} - \{0\}$.

(i) The functions $|f|$, $\alpha f$, $\alpha + f$, and $f^2$ are measurable.

(ii) If $f \neq 0$, then $1/f$ is measurable.

(iii) For any measurable subset $E'$ of $E$, the restriction $f|_{E'}$ is measurable.

Proof. The statements in (i) and (ii) are consequences of the following set identities, where all of the sets on the right are measurable by hypothesis:

$$\begin{aligned}
[|f| > c] &= \begin{cases} [f > c] \cup [f < -c] & \text{if } c \geq 0, \\ E & \text{if } c < 0, \end{cases} \\
[\alpha + f > c] &= [f > c - \alpha], \\
[\alpha f > c] &= \begin{cases} [f > c/\alpha] & \text{if } \alpha > 0, \\ [f < c/\alpha] & \text{if } \alpha < 0, \end{cases} \\
[f^2 > c] &= \begin{cases} [f > \sqrt{c}] \cup [f < -\sqrt{c}] & \text{if } c \geq 0, \\ E & \text{if } c < 0, \end{cases}, \\
\left[\frac{1}{f} > c\right] &= \begin{cases} [f > 0] \cap \left[f < \frac{1}{c}\right] & \text{if } c > 0, \\ [f > 0] & \text{if } c = 0, \\ [f > 0] \cup \left[f < \frac{1}{c}\right] & \text{if } c < 0. \end{cases}
\end{aligned}$$

To prove (iii), it suffices to recognize that

$$[f|_{E'} > c] = \{x \in E : f|_{E'}(x) > c\} = [f > c] \cap E'$$

for every $c \in \mathbb{R}$. □

Proposition 1.3. Let $f : E \to \mathbb{R}^*$ and $g : E \to \mathbb{R}$ be measurable. Then

(i) the set $[f > g] = \{x \in E : f(x) > g(x)\}$ is measurable,

(ii) the functions $f \pm g$ are measurable,

(iii) the function $fg$ is measurable, and

(iv) if $g \neq 0$, the function $f/g$ is measurable.

Proof. (i) Let $\{q_n\}$ be an enumeration of $\mathbb{Q}$.

Then

$$[f > g] = \bigcup \{[f \geq q_n] \cap [g < q_n]\}.$$

Hence $[f > g]$ is measurable.

(ii) For all $c \in \mathbb{R}$ we have $[f + g > c] = [f > -g + c]$.

Since $-g + c$ is measurable by Proposition 1.2, we have that $[f > -g + c]$ is measurable by part (i), and so $f + g$ is measurable.

Similarly, we have $f - g$ is measurable.

(iii) By Proposition 1.2, the functions $(1/4)(f+g)^2$ and $(1/4)(f-g)^2$ are measurable because $f+g$ and $f-g$ are measurable by part (ii).

Since

$$fg = \frac{(f+g)^2}{4} - \frac{(f-g)^2}{4}$$

the function $fg$ is measurable.

(iv) The function $1/g$ is measurable by Proposition 1.2.

Hence the function $f/g = f(1/g)$ is measurable by part (iii). $\square$