Math 541 Lecture #25
III.1: Measurable Functions, Part II

Proposition 1.4. Let $\{f_n\}$ be a sequence of measurable functions defined on E. Then the functions

$$\varphi = \sup f_n,\ \ \psi = \inf f_n,\ \ f'' = \limsup f_n,\ \ f' = \liminf f_n$$

on E are measurable.

Proof. For $c \in \mathbb{R}$ we have

$$\begin{aligned}
[\varphi > c] &= \{x \in E : \varphi(x) > c\} \\
&= \{x \in E : \sup_n f_n(x) > c\} \\
&= \{x \in E : f_n(x) > c \text{ for some } n\} \\
&= \bigcup_{n=1}^{\infty} \{x \in E : f_n(x) > c\} \\
&= \bigcup_{n=1}^{\infty} [f_n > c].
\end{aligned}$$

Because each $[f_n > c]$ is measurable, so then is $[\varphi > c]$, and hence φ is a measurable function.

Similarly, for $c \in \mathbb{R}$ we have

$$\begin{aligned}
[\psi \geq c] &= \{x \in E : \psi(x) \geq c\} \\
&= \{x \in E : \inf f_n(x) \geq c\} \\
&= \{x \in E : f_n(x) \geq c \text{ for all } n\} \\
&= \bigcap_{n=1}^{\infty} \{x \in E : f_n(x) \geq c\} \\
&= \bigcap_{n=1}^{\infty} [f_n \geq c].
\end{aligned}$$

Because $[f_n \geq c]$ is measurable, so then is $[\psi \geq c]$, and hence ψ is a measurable function.

By what we have shown, the functions

$$\varphi_n = \sup_{j \geq n} f_j \text{ and } \psi_n = \inf_{j \geq n} f_j,$$

are measurable, and so

$$\limsup f_n = \inf_{n \geq 1} \sup_{j \geq n} f_j \text{ and } \liminf f_n = \sup_{n \geq 1} \inf_{j \geq n} f_j$$

are also measurable functions. $\square$

Definitions. Let $\{X, \mathcal{A}, \mu\}$ be a measure space, and let $E \in \mathcal{A}$.

For two real-extended valued functions f and g defined on E, we say that f **equals** g **almost everywhere**, written $f = g$ a.e. in E, if there exists a measurable subset $\mathcal{E} \subset E$ for which $f(x) = g(x)$ for all $x \in E - \mathcal{E}$ and $\mu(\mathcal{E}) = 0$.

A property of an extended real-valued function f defined on a measure space $\{X, \mathcal{A}, \mu\}$ is said to hold almost everywhere if it holds except on a set of measure zero.

Lemma 1.5. Let $\{X, \mathcal{A}, \mu\}$ be a complete measure space. For $E \in \mathcal{A}$, if $f : E \to \mathbb{R}^*$ is measurable, and $g : E \to \mathbb{R}^*$ satisfies $f = g$ a.e. in E, then g is measurable.

Proof. Let $\mathcal{E} = [f \neq g] = \{x \in E : f(x) \neq g(x)\}$.

By hypothesis, the set $\mathcal{E}$ is measurable, has measure zero, and every subset of $\mathcal{E}$ is measurable and has measure zero.

Let $c \in \mathbb{R}$ be arbitrary.

Because $f = g$ a.e. in E we have that

$$[g > c] = \{[f > c] \cap (E - \mathcal{E})\} \cup \{[g > c] \cap \mathcal{E}\}.$$

The first set on the right is measurable because $[f > c]$ and $E - \mathcal{E}$ are measurable, and the second set on the right is measurable because it is a subset of the measurable $\mathcal{E}$ of measure zero.

Thus g is measurable. $\square$

Corollary 1.6. Let $\{X, \mathcal{A}, \mu\}$ be a complete measure space. For $E \in \mathcal{A}$, let $\{g_n\}$ be a sequence of measurable extended real-valued functions with domain E. If

$$g(x) = \lim g_n(x) \text{ exists a.e. in } E,$$

then $g : E \to \mathbb{R}^*$ is measurable.

Proof. Let $\mathcal{E}$ be the subset of E on which $\lim g_n$ does not exist, i.e., for $x \in \mathcal{E}$ we have $\lim g_n(x)$ does not exist.

Since $\lim g_n(x)$ exists a.e. in E, we have $\mathcal{E}$ is measurable and $\mu(\mathcal{E}) = 0$.

For $x \in \mathcal{E}$, define $g(x)$ arbitrarily, i.e., randomly pick the value of $g(x)$ from $\mathbb{R}^*$ for each $x \in \mathcal{E}$.

Define a function $f : E \to \mathbb{R}^*$ by $f(x) = g(x)$ for $x \in E - \mathcal{E}$, and $f(x) = \infty$ for $x \in \mathcal{E}$.

If f is indeed measurable, then by Lemma 1.5, the function g is measurable.

Let $c \in \mathbb{R}$.

Since $E = (E - \mathcal{E}) \cup \mathcal{E}$ disjointly, we have the disjoint union

$$[g_n > c] = \{x \in E - \mathcal{E} : g_n(x) > c\} \cup \{x \in \mathcal{E} : g_n(x) > c\}.$$

Then

$$\{x \in E - \mathcal{E} : g_n(x) > c\} = [g_n > c] - \{x \in \mathcal{E} : g_n(x) > c\}.$$

The set $[g_n > c]$ is measurable because g_n is measurable, and $\{x \in \mathcal{E} : g_n(x) > c\}$ is measurable because it is a subset of a measurable set of measure zero.

Thus $\{x \in E - \mathcal{E} : g_n(x) > c\}$ is measurable for every $c \in \mathbb{R}$.

We have shown that $\{g_n\}$ restricted to $E - \mathcal{E}$ is a sequence of measurable extended real-valued functions.

Since $\{g_n(x)\}$ converges for each $x \in E - \mathcal{E}$, we have

$$f(x) = g(x) = \lim g_n(x) = \limsup g_n(x), \; x \in E - \mathcal{E}.$$

By Proposition 1.4, we have the measurability of f restricted to $E - \mathcal{E}$.

Thus for any $c \in \mathbb{R}$ we have

$$\begin{aligned} [f > c] &= \{x \in E : f(x) > c\} \\ &= \{x \in E - \mathcal{E} : f(x) > c\} \cup \{x \in \mathcal{E} : f(x) > c\} \\ &= \{x \in E - \mathcal{E} : f(x) > c\} \cup \mathcal{E} \end{aligned}$$

because $f(x) = \infty$ for $x \in \mathcal{E}$.

Since each set is measurable, the union is measurable, and hence f is measurable.

By Lemma 1.5, the function $g : E \to \mathbb{R}^*$ is measurable. $\square$.