Math 541 Lecture #26
III.2: The Egorov Theorem
III.3: Approximating Measurable Functions by Simple Functions, Part I

How close to being uniform is the pointwise convergence of a sequence of real-valued measurable functions $\{f_n\}$ to a real-valued measurable function f, on a measurable E?

The Egorov Theorem gives the answer on how pointwise convergence is nearly uniform convergence when E has finite measure (see the Appendix for an example).

Theorem (Egorov). For a measurable E, suppose $\{f_n\}$ and f are measurable real-valued functions defined on E. If $\mu(E) < \infty$ and $\{f_n\}$ converges a.e. in E to f, then for every $\eta > 0$ there exists a measurable set E_η such that $E_\eta \subset E$, $\mu(E - E_\eta) \leq \eta$, and $\{f_n\}$ converges uniformly to f on E_η.

Proof. WLOG we may assume that the measurable $\{f_n\}$ converges pointwise to the measurable f on E.

For $n, k \in \mathbb{N}$, define the measurable sets

$$E_n(k) = \bigcup_{m=n}^{\infty} \left\{ x \in E : |f_m(x) - f(x)| > \frac{1}{k} \right\}.$$

Points in $E_n(k)$ are points $x \in E$ where $\{f_n(x)\}$ is not converging "too fast" to $f(x)$.

When k is fixed, the sets $E_n(k)$ are monotone decreasing in n since the union is over less sets as n grows.

For each $x \in E$, since $f_m(x) \to f(x)$ there exists $M \in \mathbb{N}$ such that $|f_m(x) - f(x)| \leq 1/k$ for all $m \geq M$.

Thus for each $x \in E$ we have $x \notin E_n(k)$ for all large enough n, so that

$$\bigcap_{n=1}^{\infty} E_n(k) = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \left\{ x \in E : |f_m(x) - f(x)| > \frac{1}{k} \right\} = \emptyset.$$

Since $E_1(k) \subset E$ we have $\mu(E_1(k)) \leq \mu(E) < \infty$, and so

$$\lim_{n \to \infty} \mu(E_n(k)) = 0.$$

For $\eta > 0$ and $k \in \mathbb{N}$ we choose $n(k, \eta)$ large enough so that

$$\mu(E_{n(k,\eta)}(k)) \leq \frac{\eta}{2^k}.$$

We set

$$A_\eta = \bigcup_{k=1}^{\infty} E_{n(k,\eta)}(k) \subset E.$$

Then

$$\mu(A_\eta) = \mu\left(\bigcup_{k=1}^{\infty} E_{n(k,\eta)}(k) \right) \leq \sum_{k=1}^{\infty} \mu(E_{n(k,\eta)}(k) \leq \sum_{m=1}^{\infty} \frac{\eta}{2^k} = \eta.$$

Set $E_\eta = E - A_\eta$.

Since $A_\eta \subset E$, we have $E - E_\eta = A_\eta$, and hence

$$\mu(E - E_\eta) = \mu(A_\eta) \leq \eta.$$

We now show that the convergence of $\{f_n\}$ to f is uniform on E_η.

For $\epsilon > 0$ choose $k_\epsilon \in \mathbb{N}$ so that $\epsilon k_\epsilon \geq 1$, and set $n_\epsilon = n(k_\epsilon, \eta)$.

Then as

$$E_{n(k_\epsilon,\eta)}(k_\epsilon) = \bigcup_{m=n(k_\epsilon,\eta)}^{\infty} \left\{ x \in E : |f_m(x) - f(x)| > \frac{1}{k_\epsilon} \right\} \subset A_\eta,$$

we have

$$E_\eta = E - A_\eta \subset E - E_{n(k_\epsilon,\eta)}(k_\epsilon) = \bigcap_{m=n(k_\epsilon,\eta)}^{\infty} \left\{ x \in E : |f_m(x) - f(x)| \leq \frac{1}{k_\epsilon} \right\}.$$

Thus for all $n \geq n_\epsilon = n(k_\epsilon, \eta)$ and all $x \in E_\eta$ we have

$$|f_n(x) - f(x)| \leq \frac{1}{k_\epsilon} \leq \epsilon$$

because $\epsilon k_\epsilon \geq 1$. □

For Lebesgue measure μ on $\mathbb{R}^N$, we can replace E_η with a closed set.

Corollary. For a Lebesgue measurable set E in $\mathbb{R}^N$, suppose $\{f_n\}$ and f are real-valued measurable functions on E. If $\mu(E) < \infty$, and $\{f_n\}$ converges to f a.e. in E, then for every $\eta > 0$ there exists a closed set C_η such that $C_\eta \subset E$, $\mu(E - C_\eta) \leq \eta$, and $\{f_n\}$ converges uniformly to f on C_η.

§3: Approximating Measurable Functions by Simple Functions. We show that the class of simple measurable functions is dense in the set of measurable functions.

Definition. A real-valued function f defined on a measurable set E is **simple** if it is measurable and if its range is a finite set.

Example. The characteristic function $f = \chi_A$ for a measurable set A in X is a simple function because

$$[f \leq c] = \begin{cases} X & \text{if } c \geq 1, \\ X - A & \text{if } 0 \leq c < 1, \\ \emptyset & \text{if } c < 0, \end{cases}$$

and the range of f is $\{0, 1\}$.

Simple functions have a "canonical form."

If $\{a_1, a_2, \ldots, a_n\}$ are the distinct values of a simple function f defined on a measurable E, then the sets

$$E_i = \{x \in E : f(x) = a_i\} = \{x \in E : f(x) \geq a_i\} \cap \{x \in E : f(x) \leq a_i\}$$

are measurable and pairwise disjoint.

The canonical form of f is

$$f = \sum_{k=1}^{n} a_i \chi_{E_i}.$$

On the other hand, given measurable sets $E_1, E_2, \ldots, E_n$ and real numbers $a_1, a_2, \ldots, a_n$, the function

$$f = \sum_{k=1}^{n} a_i \chi_{E_i}$$

is simple but not necessarily in its canonical form, unless the sets $E_1, E_2, \ldots, E_n$ are pairwise disjoint and the values $a_1, a_2, \ldots, a_n$ are distinct.

Facts: the sum and product of simple functions are simple, but the sums and products of their canonical forms are not necessarily canonical.

Proposition 3.1. For each nonnegative measurable $f : E \to \mathbb{R}^*$, there exists a sequence of simple measurable functions $\{f_n\}$ such that $f_n \leq f_{n+1}$ (monotone nondecreasing) and

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x \in E$$

(pointwise convergence).

Proof. For each $n \in \mathbb{N}$ define a function $f_n : E \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n & \text{if } f(x) \geq n, \\ \dfrac{j}{2^n} & \text{if } \dfrac{j}{2^n} \leq f(x) < \dfrac{j+1}{2^n}, \text{ for } j = 0, 1, 2, \ldots, n2^n - 1. \end{cases}$$

By construction we have $f_n \leq f_{n+1}$.

Since f is measurable, the sets

$$\left[f \geq \frac{j}{2^n} \right] - \left[f \geq \frac{j+1}{2^n} \right], \ j = 0, 1, 2, \ldots, n2^n - 1,$$

and $[f \geq n]$ are all measurable and disjoint.

Thus each f_n is simple.

Fix $x \in E$.

If $f(x) \in \mathbb{R}$, then there exists $n_0 \in \mathbb{N}$ such that $f(x) \leq n_0$, and the definition of $f_n(x)$ implies by subtracting $f_n(x) = j/2^n$ from $j/2^n \leq f(x) < (j+1)/2^n$ that

$$0 \leq f(x) - f_n(x) \leq \frac{1}{2^n} \text{ for all } n \geq n_0.$$

If $f(x) = \infty$, then $f_n(x) = n$ for all n.

In either case, we have $\lim f_n(x) = f(x)$.

Thus $\{f_n\}$ converges pointwise to $f(x)$ for each $x \in E$. □

Appendix

Example. Equip $\mathbb{R}$ with Lebesgue measure.

For each $n \in \mathbb{N}$, the function $f_n(x) = x^n$, $x \in \mathbb{R}$, is measurable because $g(x) = x$ is measurable ($[g > c] = \{x \in \mathbb{R} : x > c\} = (c, \infty]$ is measurable) and $[g(x)]^n = f_n(x)$ is measurable.

The restriction of f_n to the measurable $E = [0, 1]$ is also measurable.

Thus the sequence $\{f_n\}$ on $[0, 1]$ is a sequence of measurable real-valued functions.

The pointwise limit of $\{f_n\}$ on $[0, 1]$ is the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1, \end{cases}$$

which is measurable because

$$[f > c] = \{x \in [0, 1] : f(x) > c\} = \begin{cases} \emptyset & \text{if } c \geq 1, \\ \{1\} & \text{if } 0 \leq c < 1, \\ [0, 1] & \text{if } c < 0. \end{cases}$$

If the pointwise convergence were uniform on $[0, 1]$, then the limit function would be continuous because each f_n is continuous; but the limit function is not continuous, and so the pointwise convergence is not uniform on $[0, 1]$.

However, if we restrict to the closed interval $[0, \beta]$ for any $0 < \beta < 1$, then the pointwise convergence is uniform on $[0, \beta]$.

We can choose β close to 1 so that $\mu([0, 1] - [0, \beta]) = 1 - \beta$ is small.

Proof of Corollary of the Egorov Theorem. By the Egorov Thoerem, for each $\eta > 0$ there exists a measurable set E_η such that $E_\eta \subset E$, $\mu(E - E_\eta) \leq \eta/2$, and $\{f_n\}$ converges uniformly to f on E_η.

Since $\mu(E) < \infty$, then $\mu(E_\eta) < \infty$, and so by Proposition 15.3 there exists a closed set C_η such that $C_\eta \subset E_\eta$ and $\mu(E_\eta - C_\eta) \leq \eta/2$.

Since $E - C_\eta = (E - E_\eta) \cup (E_\eta - C_\eta)$ disjointly, we have that

$$\mu(E - C_\eta) = \mu(E - E_\eta) + \mu(E_\eta - C_\eta) \leq \eta.$$

Since $C_\eta \subset E_\eta$ and $\{f_n\}$ converges uniformly to f on E_η, we have that $\{f_n\}$ converges uniformly to f on C_η. $\square$