Math 541 Lecture #27
III.3: Approximating Measurable Functions by Simple Functions, Part II
III.5: Quasi-Continuous Functions and Lusin's Theorem

§3: Approximating Measurable Functions by Simple Functions. We show that there exists a sequence of simple measurable functions converging pointwise to a measurable function.

To apply Proposition 3.1 to a function $f : E \to \mathbb{R}^*$, we decompose f into its positive and negatives parts.

Set

$$f^+ = \frac{|f| + f}{2} \text{ and } f^- = \frac{|f| - f}{2}.$$

Then $f^+ \geq 0$, $f^- \geq 0$, and $f = f^+ - f^-$.

If f is measurable, then so are f^+ and f^-.

Corollary 3.2. For a measurable $f : E \to \mathbb{R}^*$ there exists a sequence of simple functions $\{f_n\}$ such that $f(x) = \lim f_n(x)$ for all $x \in E$.

Proof. We apply Proposition 3.2 to each of f^+ and f^-, giving sequences $\{f_n^+\}$ and $\{f_n^-\}$ of simple functions that converge pointwise respectively to f^+ and f^- on E.

Then the sequence $\{f_n\}$ defined by $f_n = f_n^+ - f_n^-$, consisting of simple functions, converges to f pointwise on E. $\square$

§5. Quasi-continuous Functions and Lusin's Theorem. We will characterize measurable functions on $\mathbb{R}^N$ with respect to Lebesgue measure.

Definition. For a Lebesgue measurable set E in $\mathbb{R}^N$, a function $f : E \to \mathbb{R}^*$ is **quasi-continuous** if for every $\epsilon > 0$ there exists a closed set $E_{c,\epsilon}$ such that $E_{c,\epsilon} \subset E$, $\mu(E - E_{c,\epsilon}) \leq \epsilon$, and the restriction of f to $E_{c,\epsilon}$ is continuous.

Proposition 5.1. A simple function defined on a bounded measurable set E in $\mathbb{R}^N$ is quasi-continuous.

Proof. Let $f : E \to \mathbb{R}$ be simple and let $\{a_1, a_2, \ldots, a_n\}$ be the range of f.

Then the sets $E_i = \{x \in E : f(x) = a_i\}$ are measurable and bounded

For $\epsilon > 0$, there exists, by Proposition 12.4 in Chapter II, closed sets $E_{c,i}$ such that

$$E_{c,i} \subset E_i, \quad \mu(E_i - E_{c,i}) \leq \epsilon/n, \text{ for } i = 1, 2, \ldots, n.$$

Then the closed (and bounded and hence compact) set

$$E_{c,\epsilon} = \bigcup_{k=1}^{n} E_{c,i}$$

satisfies

$$E - E_{c,\epsilon} = E \cap E_{c,\epsilon}^c = E \cap \left(\bigcap_{k=1}^{n} E_{c,i}^c \right) = \bigcap_{k=1}^{n} \left(E \cap E_{c,i}^c \right)$$
$$= \bigcap_{k=1}^{n} \left(E - E_{c,i} \right) \subset \bigcup_{k=1}^{n} \left(E - E_{c,i} \right).$$

Thus

$$\mu(E - E_{c,\epsilon}) \leq \sum_{k=1}^{n} \mu(E - E_{c,i}) = \sum_{k=1}^{n} \frac{\epsilon}{n} = \epsilon.$$

Because the sets $E_{c,1}, E_{c,2}, \ldots, E_{c,n}$ are compact and disjoint, they are at positive mutual distances from each other.

Thus, since f is constant on each $E_{c,i}$, the function f is continuous on $E_{c,\epsilon}$.

Thus f is quasi-continuous. □

Theorem 5.2 (Lusin). Let E be a bounded Lebesgue measurable set in $\mathbb{R}^N$. A function $f : E \to \mathbb{R}$ is measurable if and only if it is quasi-continuous.

Proof. Suppose for a bounded E that the measurable $f : E \to \mathbb{R}$ is nonnegative.

By Proposition 3.1 there exists a sequence of simple functions $\{f_n\}$ that converges pointwise to f.

By Proposition 5.1 for each $n \in \mathbb{N}$ the function f_n is quasi-continuous.

For $\epsilon > 0$ there then exists closed sets $E_{c,n}$ such that $E_{c,n} \subset E$,

$$\mu(E - E_{c,n}) \leq \frac{\epsilon}{2^{n+1}},$$

and the restriction of f_n to $E_{c,n}$ is continuous.

By the Egorov Theorem there is a closed set $E_{c,0}$ (the set $E_{c,n}$ with $n = 0$) for which $E_{c,0} \subset E$, $\mu(E - E_{c,0}) \leq \epsilon/2$, and $\{f_n\}$ converges uniformly to f on $E_{c,0}$.

The set

$$E_{c,\epsilon} = E_{c,0} \cap \bigcap_{n=1}^{\infty} E_{c,n} = \bigcap_{n=0}^{\infty} E_{c,n}$$

is a closed subset of E for which

$$E - E_{c,\epsilon} = E - \bigcap_{n=0}^{\infty} E_{c,n} = E \cap \bigcup_{n=0}^{\infty} E_{c,n}^c = \bigcup_{n=0}^{\infty} (E \cap E_{c,n}^c) = \bigcup_{n=0}^{\infty} (E - E_{c,n}).$$

Hence

$$\mu(E - E_{c,\epsilon}) = \mu \left(\bigcup_{n=0}^{\infty} (E - E_{c,n}) \right) \leq \sum_{n=0}^{\infty} \mu(E - E_{c,n}) \leq \sum_{n=0}^{\infty} \frac{\epsilon}{2^{n+1}} = \epsilon.$$

Since each f_n is continuous on $E_{c,\epsilon}$ and $\{f_n\}$ converges uniformly to f on $E_{c,\epsilon}$, the function f restricted to $E_{c,\epsilon}$ is continuous, and hence f is quasi-continuous.

For f not necessarily nonnegative, we decompose $f = f^+ - f^-$ into its positive and negative parts, and apply the above argument to f^+ and f^-.

This shows that f is the difference of two quasi-continuous functions, which is a quasi-continuous function. [You have it as a homework problem to prove this.]

Now suppose that f is quasi-continuous.

For ϵ there exists a closed set $E_{c,\epsilon}$ such that $E_{c,\epsilon} \subset E$, $\mu(E - E_{c,\epsilon}) \leq \epsilon$, and f restricted to $E_{c,\epsilon}$ is continuous.

To show that $[f \geq c]$ is measurable for all $c \in \mathbb{R}$, we have that (the disjoint union)

$$[f \geq c] = \big([f \geq c] \cap E_{c,\epsilon}\big) \cup \big([f \geq c] \cap (E - E_{c,\epsilon})\big).$$

With the restriction of f to $E_{c,\epsilon}$ being continuous, the set

$$[f \geq c] \cap E_{c,\epsilon} = \{x \in E_{c,\epsilon} : f(x) \geq c\}$$

is closed because it is the preimage of a closed set by a continuous function.

Moreover, by monotonicity of the Lebesgue outer measure, we have

$$\mu_e\big([f \geq c] \cap (E - E_{c,\epsilon})\big) \leq \mu_e(E - E_{c,\epsilon}) = \mu(E - E_{c,\epsilon}) \leq \epsilon.$$

Recall that Proposition 12.4 of Chapter II characterizes the Lebesgue measurability of bounded sets in terms of the Lebesgue outer measure μ_e: a bounded set A is Lebesgue measurable if and only if for every $\epsilon > 0$ there exists a closed set $A_{c,\epsilon}$ such that $A_{c,\epsilon} \subset A$ and $\mu_e(A - A_{c,\epsilon}) \leq \epsilon$.

For $A = [f \geq c]$ and $A_{c,\epsilon} = [f \geq c] \cap E_{c,\epsilon}$ we have

$$A - A_{c,\epsilon} = [f \geq c] - \big([f \geq c] \cap E_{c,\epsilon}\big) = [f \geq c] \cap (E - E_{c,\epsilon})$$

where $\mu_e(A - A_{c,\epsilon}) = \mu([f \geq c] \cap (E - E_{c,\epsilon})) \leq \epsilon$,

This satisfies the characterization of measurability, and so $[f \geq c]$ is measurable. $\square$

Corollary. For a bounded Lebesgue measurable set E in $\mathbb{R}^N$, every continuous function $f : E \to \mathbb{R}$ is measurable.

Proof. For a bounded Lebesgue measurable set E and every $\epsilon > 0$ there is by Proposition 12.4 a closed set $E_{c,\epsilon}$ such that $E_{c,\epsilon} \subset E$ and $\mu(E - E_{c,\epsilon}) \leq \epsilon$.

Since $f : E \to \mathbb{R}$ is continuous, the restriction of f to $E_{c,\epsilon}$ is also continuous, and hence f is quasi-continuous.

By Lusin's Theorem, the the continuous function $f : E \to \mathbb{R}$ is measurable. $\square$

Homework Problem 27A. Prove that the difference of two quasi-continuous functions is quasi-continuous.