Math 541 Lecture #28
III.6: Integral of Simple Functions
III.7: The Lebesgue Integral of Nonnegative Functions

§6: Integral of Simple Functions. For a measure space $\{X, \mathcal{A}, \mu\}$ let $E \in \mathcal{A}$.

For $A \in \mathcal{A}$ and $\alpha \in \mathbb{R}$ we define the Lebesgue integral of $\alpha\chi_A$ to be

$$\int_E \alpha\chi_A \, d\mu = \begin{cases} \alpha\mu(E \cap A) & \text{if } \alpha \neq 0, \\ 0 & \text{if } \alpha = 0. \end{cases}$$

The first case is well-defined as an element of $\mathbb{R}^*$ for all $\alpha \in \mathbb{R} - \{0\}$ because $\mu(E \cap A) \in \mathbb{R}^*$; it is possible for $\alpha\mu(E \cap A) = \infty$, and his happens when $\mu(E \cap A) = \infty$ and $\alpha \neq 0$.

The second case hides the assumption $0 \cdot \infty = 0$, i.e., $0 \cdot \mu(E \cap A) = 0$ even when $\mu(E \cap A) = \infty$.

So we may simplify the definition of the Lebesgue integral of $\alpha\chi_A$ to

$$\int_E \alpha\chi_A \, d\mu = \alpha\mu(E \cap A)$$

with the understanding that $0 \cdot \infty = 0$.

Let $f : E \to \mathbb{R}^*$ be a nonnegative simple function with the canonical representation

$$f = \sum_{i=1}^n a_i \chi_{E_i},$$

where $\{a_1, a_2, \ldots, a_n\}$ are the distinct values in the range of f and $E_1, E_2, \ldots, E_n$ are mutually disjoint measurable sets whose union is E.

The Lebesgue integral of f is defined by

$$\int_E f \, d\mu = \sum_{i=1}^n \int_E a_i \chi_{E_i} \, d\mu = \sum_{i=1}^n a_i \mu(E \cap E_i) = \sum_{i=1}^n a_i \mu(E_i).$$

This could be finite or infinite.

If the Lebesgue integral of f is finite, we say that f is Lebesgue integrable in E.

We note for a nonnegative simple $f : E \to \mathbb{R}^*$ that is integrable, that $[f > 0]$ has finite measure, for if $\mu([f > 0]) = \infty$, then for some positive image value a_i of f the measurable set E_i would have to have infinite Lebesgue measure, and hence

$$\int_E f \, d\mu = \sum_{i=1}^n a_i \mu(E_i) = \infty.$$

We have defined the Lebesgue integral of a nonnegative simple function f in terms of a canonical representation of f.

What is the Lebesgue integral of a nonnegative simple function

$$f = \sum_{i=1}^{m} b_i \chi_{F_i}$$

not assumed to be in canonical form?

Without loss of generality we may assume that the measurable sets $F_1, F_2, \ldots, F_m$ are pairwise disjoint and whose union is E.

The values $\{b_1, b_2, \ldots, b_m\}$ of f are not assumed to be distinct.

Suppose that there are n distinct image values of f, which WLOG we may assume are $b_1, b_2, \ldots, b_n$ for $n \leq m$.

We put f into canonical form by setting

$$E_i = \bigcup \{F_j : b_j = b_i\},\ i = 1, 2, \ldots, n.$$

Then

$$f = \sum_{i=1}^{n} b_i \chi_{E_i}$$

is a canonical representation of f for which

$$\int_E f\ d\mu = \sum_{i=1}^{n} b_i \mu(E_i) = \sum_{i=1}^{n} b_i \sum_{\{j : b_j = b_i\}} \mu(F_j) = \sum_{j=1}^{m} b_j \mu(F_j).$$

Thus the integral is independent of the representation of the the nonnegative simple function.

The following two properties of Lebesgue integration are Homework problems.

(1) For nonnegative simple functions $f, g : E \to \mathbb{R}^*$, if $f \leq g$ a.e. in E, then

$$\int_E f\ d\mu \leq \int_E g\ d\mu.$$

(2) For Lebesgue integrable nonnegative simple functions $f, g : E \to \mathbb{R}^*$, we have

$$\int_E (\alpha f + \beta g) d\mu = \alpha \int_E f\ d\mu + \beta \int_E g\ d\mu$$

for all $\alpha, \beta \in \mathbb{R}$.

§7: The Lebesgue Integral of Nonnegative Functions.

For a measurable nonnegative function $f : E \to \mathbb{R}^*$, let $\mathcal{S}_f$ denote the collection of all nonnegative simple functions $\zeta : E \to \mathbb{R}$ such that $\zeta \leq f$.

Since $\zeta = 0$ satisfies $\zeta \leq f$, the collection $\mathcal{S}_f$ is nonempty.

The Lebesgue integral of f over E is defined to be

$$\int_E f\ d\mu = \sup_{\zeta \in \mathcal{S}_f} \int_E \zeta\ d\mu.$$

The supremum could be finite or infinite, but it always exists.

What is the difference between the Lebesgue integral and the Riemann integral? In the latter we partition the domain of f while in the former we partition the range of f.

A nonnegative measurable function $f : E \to \mathbb{R}^*$ is Lebesgue integrable if

$$\int_E f \; d\mu < \infty.$$

For example, if μ is the counting measure on $\mathbb{N}$, then a nonnegative simple function $f : \mathbb{N} \to [0, \infty]$ (a extended real-valued nonnegative sequence) is Lebesgue integrable if and only if the series of nonnegative terms

$$\int_E f \; d\mu = \sup_{\zeta \in \mathcal{S}_f} \int_E \zeta \; d\mu = \sum_{n=1}^{\infty} f(n)$$

converges to a real number (where the supremum is realized by the limit of the sequence of partial sums of the series).

For a measurable nonpositive function $f : E \to \mathbb{R}$ we define

$$\int_E f \; d\mu = -\int_E (-f) d\mu \quad (f \leq 0).$$

If $f, g : E \to \mathbb{R}^*$ are measurable and nonnegative with $f \leq g$ a.e. in E, then $\mathcal{S}_f \subset \mathcal{S}_g$, so that

$$\int_E f \; d\mu \leq \int_E g \; d\mu.$$

A measurable function $f : E \to \mathbb{R}^*$ is Lebesgue integrable if $|f|$ is Lebesgue integrable.

From the decomposition $f = f^+ - f^-$ where

$$f^+ = \frac{|f| + f}{2}, \quad f^- = \frac{|f| - f}{2},$$

we have, since $f \leq |f|$ and $-f \leq |f|$ that

$$f^+ \leq \frac{|f| + |f|}{2} = |f|, \quad f^- \leq \frac{|f| + |f|}{2} = |f|.$$

Thus if f is Lebesgue integrable, then f^+ and f^- are Lebesgue integrable because

$$\int_E f^+ \; d\mu \leq \int_E |f| d\mu < \infty, \quad \int_E f^- \; d\mu \leq \int_E |f| d\mu < \infty.$$

For a Lebesgue integrable function $f : E \to \mathbb{R}^*$ we set

$$\int_E f \; d\mu = \int_E f^+ \; d\mu - \int_E f^- \; d\mu.$$

If $E' \subset E$ is measurable and $f : E \to \mathbb{R}^*$ is Lebesgue integrable, then $f\chi_{E'}$ (the restriction of f to E') is Lebesgue integrable and

$$\int_{E'} f \, d\mu = \int_E f\chi_{E'} \, d\mu.$$

For a measurable $f : E \to \mathbb{R}^*$ we set

$$\int_E f \, d\mu = \infty \text{ if } \int_E f^+ \, d\mu = \infty \text{ and } \int_E f^- \, d\mu < \infty,$$

and

$$\int_E f \, d\mu = -\infty \text{ if } \int_E f^+ \, d\mu < \infty \text{ and } \int_E f^- \, d\mu = \infty.$$

We leave

$$\int_E f \, d\mu \text{ undefined if } \int_E f^+ \, d\mu = \infty \text{ and } \int_E f^- \, d\mu = \infty.$$