Math 541 Lecture #29
III.8: Fatou's Lemma and the Monotone Convergence Theorem

§8: Fatou's Lemma and the Monotone Convergence Theorem. We will present these results in a manner that differs from the book: we will first prove the Monotone Convergence Theorem, and use it to prove Fatou's Lemma.

Proposition. Let $\{X, \mathcal{A}, \mu\}$ be a measure space. For $E \in \mathcal{A}$, if $\varphi : E \to \mathbb{R}$ is a nonnegative simple function, then

$$A \to \int_A \varphi \, d\mu, \ A \in \mathcal{A},$$

is a measure on the σ-algebra $\mathcal{A}$.

Proof. If $\varphi = \sum_{i=1}^n b_i \chi_{B_i}$ is the canonical representation of the nonnegative simple φ, then

$$\int_A \varphi \, d\mu = \sum_{i=1}^n b_i \mu(A \cap B_i).$$

The domain of $A \to \int_A \varphi \, d\mu$ is the σ-algebra $\mathcal{A}$, and $A \to \int_A \varphi \, d\mu$ is nonnegative for each A because $b_i \geq 0$ and $\mu(A \cap B_i) \geq 0$ for all $i = 1, 2, \ldots, n$.

For $A = \emptyset$, we have $\int_A \varphi \, d\mu = 0$.

For a countable collection $\{A_k\}$ of pairwise disjoint sets in $\mathcal{A}$, we have for $A = \cup A_k$ that

$$\begin{aligned}
\int_A \varphi \, d\mu &= \sum_{i=1}^n a_i \mu(A \cap B_i) \\
&= \sum_{i=1}^n \mu \left(\left(\bigcup_{k=1}^\infty A_k \right) \cap B_i \right) \\
&= \sum_{i=1}^n \sum_{k=1}^\infty a_i \mu(A_k \cap B_i) \\
&= \sum_{k=1}^\infty \sum_{i=1}^n a_i \mu(A_k \cap B_i) \\
&= \sum_{k=1}^\infty \int_{A_k} \varphi \, d\mu.
\end{aligned}$$

This gives countable additivity. $\square$

The Monotone Convergence Theorem. If $\{f_n\}$ is a sequence of nonnegative measurable functions defined on a measurable E, such that $f_n \leq f_{n+1}$ for all n, and $f = \lim f_n = \sup f_n$ (which exists), then

$$\int_E f \, d\mu = \lim_{n \to \infty} \int_E f_n \, d\mu.$$

Proof. Because $\{f_n\}$ is monotone nondecreasing, the sequence

$$\left\{\int_E f_n \, d\mu\right\}$$

is monotone nondecreasing and so its limit exists (possibly equal to ∞).

Since $f_n \leq \sup f_n = f$, we have for all n that

$$\int_E f_n \, d\mu \leq \int_E f \, d\mu,$$

and hence that

$$\lim_{n\to\infty} \int_E f_n \, d\mu \leq \int_E f \, d\mu.$$

To establish the reverse inequality, we fix $\alpha \in (0,1)$, let ψ be a simple function satisfying $0 \leq \psi \leq f$, and set

$$E_n = [f_n \geq \alpha\psi] = \{x \in E : f_n(x) \geq \alpha\psi(x)\}.$$

Then $\{E_n\}$ is monotone increasing sequence of measurable sets whose union is E, and we have

$$\int_E f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \int_{E_n} \alpha\psi \, d\mu = \alpha \int_{E_n} \psi \, d\mu.$$

By the Proposition, the function

$$\nu(A) = \int_A \psi \, d\mu, \quad A \in \mathcal{A}$$

is a measure on $\mathcal{A}$.

Since $\{E_n\}$ is monotone increasing with $\cup E_n = E$, we have that

$$\lim_{n\to\infty} \int_{E_n} \psi \, d\mu = \lim_{n\to\infty} \nu(E_n) = \nu(E) = \int_E \psi \, d\mu.$$

Thus

$$\lim_{n\to\infty} \int_E f_n \, d\mu \geq \alpha \lim_{n\to\infty} \int_{E_n} \psi \, d\mu = \alpha \int_E \psi \, d\mu.$$

As this holds for all $\alpha \in (0,1)$, it holds also for $\alpha = 1$, and taking the supremum over all simple $0 \leq \psi \leq f$ gives

$$\lim_{n\to\infty} \int_E f_n \, d\mu \geq \int_E f \, d\mu.$$

This is the reverse inequality sought. □

Fatou's Lemma. If $\{f_n\}$ is a sequence of nonnegative measurable functions on E, then

$$\int_E \liminf f_n \, d\mu \leq \liminf \int_E f_n \, d\mu.$$

Proof. For each k we have for all $j \geq k$ that

$$\inf_{n \geq k} f_n \leq f_j.$$

Hence for all $j \geq k$ we have

$$\int_E \inf_{n \geq k} f_n \, d\mu \leq \int_E f_j \, d\mu.$$

The left-hand side is a lower bound for the integrals on the right-hand side, so that

$$\int_E \inf_{n \geq k} f_n \, d\mu \leq \inf_{j \geq k} \int_E f_j \, d\mu.$$

Since the sequence $\{\inf_{n \geq k} f_n\}$ is monotone nondecreasing in k, we apply the Monotone Convergence Theorem to get

$$\begin{aligned}
\int_E \liminf_{n \to \infty} f_n \, d\mu &= \int_E \lim_{k \to \infty} \inf_{n \geq k} f_n \, d\mu \\
&= \lim_{k \to \infty} \int_E \inf_{n \geq k} f_n \, d\mu \\
&\leq \lim_{k \to \infty} \inf_{j \geq k} \int_E f_j \, d\mu \\
&= \liminf_{n \to \infty} \int_E f_n \, d\mu,
\end{aligned}$$

giving the desired result.