Math 541 Lecture #30
III.9: Basic Properties of the Lebesgue Integral

§9: Basic Properties of the Lebesgue Integral. The Lebesgue integral has many of the properties of the Riemann integral.

Proposition 9.1. Let $f, g : E \to \mathbb{R}$ be Lebesgue integrable. Then (1) for all $\alpha, \beta \in \mathbb{R}$ we have

$$\int_E (\alpha f + \beta g) d\mu = \alpha \int_E f \ d\mu + \beta \int_E g \ d\mu,$$

(2) if $f \geq g$ a.e. in E, then

$$\int_E f \ d\mu \geq \int_E g \ d\mu,$$

(3) we have

$$\left| \int_E f \ d\mu \right| \leq \int_E |f| \ d\mu,$$

and (4) if E' is a measurable subset of E, then

$$\int_E f \ d\mu = \int_{E-E'} f \ d\mu + \int_{E'} f \ d\mu.$$

Proof. (1) For $\alpha \geq 0$ and $f \geq 0$, denote by $\alpha \mathcal{S}_f$ the collection of functions of the form $\alpha\zeta$ for $\zeta \in \mathcal{S}_f$.

Then $\alpha \mathcal{S}_f = \mathcal{S}_{\alpha f}$ because (1) for $\alpha\zeta \in \alpha\mathcal{S}_f$ we have $0 \leq \zeta \leq f$, so $0 \leq \alpha\zeta \leq \alpha f$, and hence $\alpha\zeta \in \mathcal{S}_{\alpha f}$, and (2) for $\varphi \in \mathcal{S}_{\alpha f}$ we have $0 \leq \varphi \leq \alpha f$ which implies (a) $\varphi = 0$ if $\alpha = 0$ and hence that $\alpha\varphi \in \alpha\mathcal{S}_f$, or (b) that $0 \leq \alpha^{-1}\varphi \leq f$ if $\alpha > 0$ and hence that $\varphi = \alpha(\alpha^{-1}\varphi) \in \alpha\mathcal{S}_f$.

Thus

$$\int_E \alpha f \ d\mu = \sup_{\eta \in \mathcal{S}_{\alpha f}} \int_E \eta \ d\mu = \sup_{\eta \in \alpha\mathcal{S}_f} \int_E \eta \ d\mu = \alpha \sup_{\zeta \in \mathcal{S}_f} \int_E \zeta \ d\mu = \alpha \int_E f \ d\mu.$$

For $\alpha < 0$ we make use of

$$\int_E f \ d\mu = - \int_E (-f) \ d\mu$$

to obtain for every nonnegative measurable $f : E \to \mathbb{R}^*$ that

$$\int_E \alpha f \ d\mu = - \int_E (-\alpha) f \ d\mu = -(-\alpha) \int_E f \ d\mu = \alpha \int_E f \ d\mu.$$

For $\alpha \geq 0$ and f of variable sign, we make use of the decomposition

$$\alpha f = (\alpha f)^+ - (\alpha f)^- = \alpha(f^+) - \alpha(f^-)$$

to obtain

$$\begin{aligned}
\int_E \alpha f \; d\mu &= \int_E (\alpha f)^+ \; d\mu - \int_E (\alpha f)^- \; d\mu \\
&= \alpha \int_E f^+ \; d\mu - \alpha \int_E f^- \; d\mu \\
&= \alpha \left(\int_E f^+ \; d\mu - \int_E f^- \; d\mu \right) \\
&= \alpha \int_E f \; d\mu.
\end{aligned}$$

For $\alpha < 0$ and f of variable sign, we make use of the decomposition

$$\alpha f = -(-\alpha f) = -\{(-\alpha f)^+ - (-\alpha f)^-\} = -\{(-\alpha) f^+ - (-\alpha) f^-\}$$

to obtain

$$\int_E \alpha f \; d\mu = -\int_E (-\alpha f) \; d\mu = -(-\alpha) \int_E f \; d\mu = \alpha \int_E f \; d\mu.$$

Thus it suffices to prove the linearity of the Lebesgue integral when $\alpha = \beta = 1$, i.e.,

$$\int_E (f + g) \; d\mu = \int_E f \; d\mu + \int_E g \; d\mu.$$

First assume that both f and g are nonnegative.

Then there exist monotone nondecreasing sequences of simple functions $\{\zeta_n\}$ and $\{\xi_n\}$ converging pointwise to f and g respectively.

Then the sequence $\{\zeta_n + \xi_n\}$ converges pointwise to $f + g$ on E, and is monotone nondecreasing because $\zeta_n + \xi_n \le \zeta_{n+1} + \xi_{n+1}$.

By the Monotone Convergence Theorem and the property of Lebesgue integration of sums of simple functions (the integral of a sum of simple functions is the sum of the integrals of the simple functions),

$$\begin{aligned}
\int_E (f + g) \; d\mu &= \lim_{n\to\infty} \int_E (\zeta_n + \xi_n) \; d\mu \\
&= \lim_{n\to\infty} \int_E \zeta_n \; d\mu + \lim_{n\to\infty} \int_E \xi_n \; d\mu \\
&= \int_E f \; d\mu + \int_E g \; d\mu.
\end{aligned}$$

Next, for $f \ge 0$ and $g \le 0$, we observe that $f + g$ is integrable because

$$|f + g| \le |f| + |g|$$

with $|f|$ and $|g|$ integrable nonnegative functions whose sum is integrable by the above argument, i.e., $\int_E (|f + g|) \; d\mu \le \int_E (|f| + |g|) \; d\mu = \int_E |f| \; d\mu + \int_E |g| \; d\mu < \infty$.

From the decomposition

$$f + g = (f + g)^+ - (f + g)^-$$

we have the decomposition

$$(f + g)^+ - g = (f + g)^- + f.$$

This together with the integral of the sum of two nonnegative functions being the sum of the integrals (proven above) gives

$$\begin{aligned}
\int_E (f + g)^+ \; d\mu + \int_E (-g) \; d\mu &= \int_E \{(f + g)^+ - g\} \; d\mu \\
&= \int_E \{(f + g)^- + f\} \; d\mu \\
&= \int_E (f + g)^- \; d\mu + \int_E f \; d\mu.
\end{aligned}$$

Since f, g, and $f + g$ are integrable, so are $-g$, $(f + g)^+$, and $(f + g)^-$, and hence

$$\begin{aligned}
\int_E (f + g) \; d\mu &= \int_E (f + g)^+ \; d\mu - \int_E (f + g)^- \; d\mu \\
&= \int_E f \; d\mu - \int_E (-g) \; d\mu \\
&= \int_E f \; d\mu + \int_E g \; d\mu.
\end{aligned}$$

This gives the result when $f \geq 0$ and $g \leq 0$.

We now apply this to Lebesgue integrable $f, g : E \to \mathbb{R}^*$ to get

$$\begin{aligned}
\int_E (f + g) \; d\mu &= \int_E \{(f^+ - f^-) + (g^+ - g^-)\} \; d\mu \\
&= \int_E \{(f^+ + g^+) - (f^- + g^-)\} \; d\mu \\
&= \int_E (f^+ + g^+) \; d\mu - \int_E (f^- + g^-) \; d\mu \\
&= \int_E f^+ \; d\mu + \int_E g^+ \; d\mu - \int_E f^- \; d\mu - \int_E g^- \; d\mu \\
&= \int_E f \; d\mu + \int_E g \; d\mu.
\end{aligned}$$

(2) With $f \geq g$ a.e. in E, we have $f - g \geq 0$ a.e. in E.

Since the Lebesgue integral contributes 0 on sets of measure zero, we may WLOG assume that $f - g \geq 0$ everywhere in E.

With $f + g$ being a Lebesgue integrable nonnegative function, its integral is nonnegative.

By the linearity of Lebesgue integration established by part (1) we have

$$0 \leq \int_E (f+g)\ d\mu = \int_E f\ d\mu + \int_E g\ d\mu,$$

which implies that

$$\int_E g\ d\mu \leq \int_E f\ d\mu.$$

(iii) Since $-|f| \leq f \leq |f|$ we have by part (2) that

$$-\int_E |f|\ d\mu \leq \int_E f\ d\mu \leq \int_E |f|\ d\mu.$$

This implies that

$$\left|\int_E f\ d\mu\right| \leq \int_E |f|\ d\mu.$$

(4) For a measurable $E' \subset E$, we have by writing $f = f\chi_{E-E'} + f\chi_{E'}$ and part (1) that

$$\begin{aligned}\int_E f\ d\mu &= \int_E f\chi_{E-E'}\ d\mu + \int_E f\chi_{E'}\ d\mu \\ &= \int_{E-E'} f\ d\mu + \int_{E'} f\ d\mu.\end{aligned}$$

This completes the proof. $\square$