Math 541 Lecture #31
III.10: Convergence Theorems
III.11: Absolute Continuity of the Integral

On a measure space $\{X, \mathcal{A}, \mu\}$, Fatou's Lemma and the Monotone Convergence Theorem hold in other situations.

Proposition 10.1. For $E \in \mathcal{A}$, let $g : E \to \mathbb{R}^*$ be integrable. For a sequence $\{f_n\}$ of extended real-valued measurable functions on E, if $f_n \geq g$ a.e. in E for all $n \in \mathbb{N}$, then

$$\int_E \liminf f_n \, d\mu \leq \liminf \int_E f_n \, d\mu.$$

Proof. By hypothesis, the measurable functions $\{f_n - g\}$ are nonnegative.

Applying Fatou's Lemma to the sequence $\{f_n - g\}$ gives

$$\int_E \liminf (f_n - g) \, d\mu \leq \liminf \int_E (f_n - g) \, d\mu.$$

The left-hand side of this is

$$\begin{aligned}\int_E \liminf (f_n - g) \, d\mu &= \int_E \{(\liminf f_n) - g\} \, d\mu \\ &= \int_E \liminf f_n \, d\mu - \int_E g \, d\mu,\end{aligned}$$

while the right-hand side is

$$\begin{aligned}\liminf \int_E (f_n - g) \, d\mu &= \liminf \left(\int_E f_n d\mu - \int_E g \, d\mu \right) \\ &= \liminf \int_E f_n \, d\mu - \int_E g \, d\mu.\end{aligned}$$

Thus

$$\int_E \liminf f_n \, d\mu - \int_E g \, d\mu \leq \liminf \int_E f_n \, d\mu - \int_E g \, d\mu$$

which implies because g is Lebesgue integrable that

$$\int_E \liminf f_n \, d\mu \leq \liminf \int_E f_n \, d\mu,$$

giving the result. □

Proposition 10.2. For a sequence $\{f_n\}$ of measurable nonnegative functions on $E \in \mathcal{A}$, there holds

$$\sum_{n=1}^{\infty} \int_E f_n \, d\mu = \int_E \left(\sum_{n=1}^{\infty} f_n \right) d\mu.$$

Proof. The sequence of partial sums

$$g_n = \left\{ \sum_{i=1}^{n} f_i \right\}$$

is a monotone nondecreasing sequence of measurable nonnegative functions.

By the Monotone Convergence Theorem we have

$$\lim_{n\to\infty} \int_E g_n \; d\mu = \int_E \lim_{n\to\infty} g_n \; d\mu.$$

For the left-hand side we have

$$\begin{aligned}
\lim_{n\to\infty} \int_E g_n \; d\mu &= \lim_{n\to\infty} \int_E \sum_{i=1}^{n} f_i \; d\mu \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \int_E f_i \; d\mu \\
&= \sum_{i=1}^{\infty} \int_E f_i \; d\mu,
\end{aligned}$$

while for the right-hand side we have

$$\begin{aligned}
\int_E \lim_{n\to\infty} g_n d\mu &= \int_E \left(\lim_{n\to\infty} \sum_{i=1}^{n} f_i \right) \; d\mu \\
&= \int_E \left(\sum_{i=1}^{\infty} f_i \right) \; d\mu.
\end{aligned}$$

This gives the result. □

Now for one of the most useful and far-reaching convergence results for Lebesgue integration.

Theorem 10.3 (Lebesgue's Dominated Convergence Theorem). Let $\{f_n\}$ be a sequence of Lebesgue integrable functions on E that converges pointwise to f, i.e.,

$$f(x) = \lim_{n\to\infty} f_n(x), \; x \in E.$$

If there is a Lebesgue integrable function $g : E \to \mathbb{R}^*$ such that

$$|f_n| \leq g \text{ a.e. in } E \text{ for all } n \in \mathbb{N}$$

(the sequence $\{f_n\}$ is *dominated* by g), then the limit function f is Lebesgue integrable and

$$\lim_{n\to\infty} \int_E f_n \; d\mu = \int_E \lim_{n\to\infty} f_n \; d\mu.$$

Proof. The limit function is measurable because $\lim f_n = \limsup f_n$.

Since $f = \lim f_n$, then $|f| = \lim |f_n|$.

By Fatou's Lemma and $|f_n| \leq g$ a.e. in E, we have

$$\begin{aligned}
\int_E |f| \; d\mu &= \int_E \lim |f_n| \; d\mu \\
&= \int_E \liminf |f_n| \; d\mu \\
&\leq \liminf \int_E |f_n| \; d\mu \\
&\leq \liminf \int_E |g| \; d\mu \\
&= \int_E |g| \; d\mu < \infty
\end{aligned}$$

Thus the limit function f is Lebesgue integrable.

Since $-f_n \leq |f_n| \leq g$ and $f_n \leq |f_n| \leq g$ we have that

$$g + f_n \geq 0 \text{ and } g - f_n \geq 0 \text{ a.e. in } E.$$

Applying Fatou's Lemma to the sequences $\{g + f_n\}$ and $\{g - f_n\}$ gives

$$\begin{aligned}
\int_E g \; d\mu + \int_E f \; d\mu &= \int_E \liminf (g + f_n) \; d\mu \\
&\leq \liminf \int_E (g + f_n) \; d\mu \\
&= \int_E g \; d\mu + \liminf \int_E f_n \; d\mu, \\
\int_E g \; d\mu - \int_E f \; d\mu &= \int_E \liminf (g - f_n) \; d\mu \\
&\leq \liminf \int_E (g - f_n) \; d\mu \\
&= \int_E g \; d\mu - \limsup \int_E f_n \; d\mu.
\end{aligned}$$

Since g is integrable, its integral is finite and cancels to give

$$\begin{aligned}
\int_E f \; d\mu &\leq \liminf \int_E f_n \; d\mu, \\
-\int_E f \; d\mu &\leq -\limsup \int_E f_n \; d\mu.
\end{aligned}$$

This imply that

$$\limsup \int_E f_n \; d\mu \leq \int_E f \; d\mu \leq \liminf \int_E f_n \; d\mu.$$

Since $\liminf \int_E f_n \, d\mu \leq \limsup \int_E f_n \, d\mu$, we have that

$$\int_E f \, d\mu \leq \liminf \int_E f_n \, d\mu \leq \limsup \int_E f_n \, d\mu \leq \int_E f \, d\mu,$$

and so

$$\liminf \int_E f_n \, d\mu = \limsup \int_E f_n \, d\mu.$$

Therefore,

$$\lim \int_E f_n \, d\mu \text{ exists and equals } \int_E f \, d\mu = \int_E \lim f_n \, d\mu.$$

This completes the proof. □

§11: Absolute Continuity of the Integral. For an integrable function $f : E \to \mathbb{R}^*$, does the value of $\int_{\mathcal{E}} |f| \, d\mu$ go to zero as $\mu(\mathcal{E})$ goes to zero?

Theorem 11.1 (Vitali). Let E be measurable, and $f : E \to \mathbb{R}^*$ be integrable. For every $\epsilon > 0$ there exists $\delta > 0$ such that for every measurable subset $\mathcal{E}$ of E satisfying $\mu(\mathcal{E}) < \delta$, we have

$$\int_{\mathcal{E}} |f| \, d\mu < \epsilon.$$

Proof. Since we are integrating $|f|$, we may assume that $f \geq 0$.

For $n \in \mathbb{N}$, define a sequence $\{f_n\}$ by

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) < n, \\ n & \text{if } f(x) \geq n. \end{cases}$$

Then $\{f_n\}$ is monotone nondecreasing sequence of measurable functions, bounded above by f, that converges pointwise to f.

Thus for each $n \in \mathbb{N}$ we have

$$\int_E f_n \, d\mu \leq \int_E f \, d\mu.$$

And by the Monotone Convergence Theorem we also have

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

Since f is integrable, we have $\int_E f \, d\mu < \infty$.

Thus for $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$\int_E f_{n_\epsilon} \, d\mu > \int_E f \, d\mu - \frac{\epsilon}{2}.$$

This says that

$$\int_E (f - f_{n_\epsilon}) \, d\mu < \frac{\epsilon}{2}.$$

Choose

$$\delta = \frac{\epsilon}{2n_\epsilon}.$$

Then for every measurable subset $\mathcal{E}$ of E satisfying $\mu(\mathcal{E}) < \delta$ we have

$$\begin{aligned}
\int_{\mathcal{E}} f \, d\mu &= \int_{\mathcal{E}} (f_{n_\epsilon} + f - f_{n_\epsilon}) \, d\mu \\
&= \int_{\mathcal{E}} f_{n_\epsilon} \, d\mu + \int_{\mathcal{E}} (f - f_{n_\epsilon}) \, d\mu \\
&\leq \int_{\mathcal{E}} n_\epsilon \, d\mu + \int_{E} (f - f_{n_\epsilon}) \, d\mu \\
&\leq n_\epsilon \mu(\mathcal{E}) + \frac{\epsilon}{2} \\
&\leq n_\epsilon \left(\frac{\epsilon}{2n_\epsilon} \right) + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

This gives the result. $\square$