Math 541 Lecture #32
III.12: Products of Measures
III.14: The Fubini-Tonelli Theorem

§12: Products of Measures. Let $\{X, \mathcal{A}, \mu\}$ and $\{Y, \mathcal{B}, \nu\}$ be two measure spaces.

For $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the product $A \times B$ is called a measurable rectangle of $X \times Y$.

The collection of measurable rectangles in $X \times Y$ is denoted by $\mathcal{R}_0$.

The intersection of two measurable rectangles $A_1 \times B_1$ and $A_2 \times B_2$ satisfies

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2),$$

and so the intersection is another measurable rectangle.

The difference of two measurable rectangles satisfies

$$(A_2 \times B_2) - (A_1 \times B_1) = ((A_2 - A_1) \times B_2) \cup ((A_1 \cap A_2) \times (B_2 - B_1)),$$

and so the difference is a disjoint union of finitely many measurable rectangles.

Thus $\mathcal{R}_0$ is a semialgebra.

Since $X \times Y \in R_0$, the semiaglebra $\mathcal{R}_0$ is also a sequential covering of $X \times Y$.

We define a nonnegative function λ on $\mathcal{R}_0$ by

$$\lambda(A \times B) = \mu(A)\nu(B).$$

Is λ a measure on $\mathcal{R}_0$?

Proposition 12.1. Let $\{A_n \times B_n\}$ be a countable collection of pairwise disjoint measurable rectangles whose union is a measurable rectangle $A \times B$. Then

$$\lambda(A \times B) = \sum \lambda(A_n \times B_n).$$

Proof. For each fixed $x \in A$ set

$$J_x = \{j \in \mathbb{N} : (x, y) \in A_j \times B_j \text{ for some } y \in B\}.$$

Since $\{A_n \times B_n\}$ is pairwise disjoint, so then is $\{A_j \times B_j : j \in J_x\}$ for each $x \in A$.

This implies that the collection $\{B_j : j \in J_x\}$ is pairwise disjoint; for suppose not, then there exist $y \in B$ and $j_1, j_2 \in J_x$ with $j_1 \neq j_2$ such that $y \in B_{j_1} \cap B_{j_2}$, whence $(x, y) \in (A_{j_1} \times B_{j_1}) \cap (A_{j_2} \times B_{j_2})$, a contradiction.

On the other hand, since $A \times B = \cup(A_n \times B_n)$, we have for each $y \in B$ that $(x, y) \in A_j \times B_j$ for some $j \in J_x$, hence

$$B = \bigcup_{j \in J_x} B_j.$$

Thus by countable additivity of ν we have for each $x \in A$ that

$$\nu(B) = \sum_{j \in J_x} \nu(B_j).$$

Since for each $y \in B$ there exists $j \in J_x$ such that $(x, y) \in A_j \times B_j$, then $\chi_{A_j}(x) = 1$ for all $j \in J_x$.

Also, since $x \in A_j \subset \cup A_n = A$, we have $\chi_A(x) = 1$.

Hence

$$\sum_{j \in J_x} \nu(B_j)\chi_{A_j}(x) = \sum_{j \in J_x} \nu(B_j) = \nu(B) = \nu(B)\chi_A(x).$$

Now $n \in J_x$ if and only if $\chi_{A_n}(x) = 1$ because when $n \in J_x$ we have $(x, y) \in A_n \times B_n$ for some $y \in B$, whence $\chi_{A_n}(x) = 1$, while if $\chi_{A_n}(x) = 1$, then $x \in A_n$, whence $(x, y) \in A_n \times B_n$ for some $y \in B_n \subset B$.

[Note: it might be that $B_n = \emptyset$, so that $A_n \times B_n = \emptyset$, but these can be removed from the pairwise disjoint collection whose union is $A \times B$ when $A \times B \neq \emptyset$, so that WLOG we may assume that each A_n and each B_n is nonempty.]

Thus

$$\sum_n \nu(B_n)\chi_{A_n}(x) = \sum_{j \in J_x} \nu(B_j)\chi_{A_j}(x),$$

and so

$$\nu(B)\chi_A(x) = \sum_n \nu(B_n)\chi_{A_n}(x).$$

Each function $f_n(x) = \nu(B)\chi_{A_n}(x)$ is a nonnegative measurable function, and we apply Proposition 10.2 to the series $\sum f_n$ to get

$$\begin{aligned}
\nu(B)\mu(A) &= \int_A \left(\sum_n \nu(B_n)\chi_{A_n}(x) \right) d\mu \\
&= \sum_n \int_A \nu(B_n)\chi_{A_n}(x) d\mu \\
&= \sum_n \nu(B_n)\mu(A_n).
\end{aligned}$$

Since $\lambda(A \times B) = \mu(A)\nu(B)$ and $\lambda(A_n \times B_n) = \mu(A_n)\nu(B_n)$, we have obtained

$$\lambda(A \times B) = \sum_n \mu(A_n)\nu(B_n) = \sum_n \lambda(A_n \times B_n).$$

Therefore, λ is a measure on the semialgebra $\mathcal{R}_0$. □

We use Theorem 11.1 in Chapter 3 to extend the measure λ on $\mathcal{R}_0$ to a complete measure, denoted by $\mu \times \nu$, on a σ-algebra, denoted by $\mathcal{A} \times \mathcal{B}$, in $X \times Y$.

The σ-algebra by $\mathcal{A} \times \mathcal{B}$ is in general not the smallest σ-algebra containing $\mathcal{R}_0$.

Theorem 12.1. Every pair of measure spaces $\{X, \mathcal{A}, \mu\}$ and $\{Y, \mathcal{B}, \nu\}$ generates a complete product measure space

$$\{X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu\}$$

where $\mathcal{A} \times \mathcal{B}$ is a σ-algebra containing $\mathcal{R}_0$, and $\mu \times \nu$ is a measure on $\mathcal{A} \times \mathcal{B}$ for which $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$ for every $A \times B \in \mathcal{R}_0$.

§14: The Theorem of Fubini-Tonelli. We can iterate Lebesgue integration on product measure spaces when certain conditions are met. We present two theorems without proof.

Theorem 14.1 (Fubini). Let $\{X, \mathcal{A}, \mu\}$ and $\{Y, \mathcal{B}, \nu\}$ be complete measure spaces. If a measurable $f : X \times Y \to \mathbb{R}^*$ is integrable, i.e.,

$$\int_{X\times Y} |f(x,y)| \, d(\mu \times \nu) < \infty,$$

then the function

$$x \to f(x,y)$$

is μ-integrable in X for ν-almost all $y \in Y$, the function

$$y \to f(x,y)$$

is ν-integrable in Y for almost all $x \in X$, the function

$$x \to \int_Y f(x,y) \, d\nu$$

is μ-integrable in X, the function

$$y \to \int_X f(x,y) \, d\mu$$

is ν-integrable in Y, and

$$\int_X \left(\int_Y f(x,y) \, d\nu \right) d\mu = \int_{X\times Y} f(x,y) \, d(\mu \times \nu) = \int_Y \left(\int_X f(x,y) \, d\mu \right) d\nu.$$

Theorem 14.2 (Tonelli). Let $\{X, \mathcal{A}, \mu\}$ and $\{Y, \mathcal{B}, \nu\}$ be complete σ-finite measure spaces. For a nonnegative measurable $f : X \times Y \to \mathbb{R}^*$, the function

$$x \to \int_Y f(x,y) \, d\nu$$

is μ-integrable in X, the function

$$y \to \int_X f(x,y) \, d\mu$$

is ν-integrable in Y, and

$$\int_X \left(\int_Y f(x,y) \, d\nu \right) d\mu = \int_{X\times Y} f(x,y) \, d(\mu \times \nu) = \int_Y \left(\int_X f(x,y) \, d\mu \right) d\nu.$$

We often use Tonelli's Theorem to establish the integrability of $f : X \times Y \to \mathbb{R}^*$ by applying either of the iterated integrals to $|f|$ to see if we get a finite value.

For those f for which either (and hence both) of the iterated integrals of $|f|$ is finite, we can then use Fubini's Theorem to evaluate the integral of f over $X \times Y$ by either of the iterated integrals.