Math 541 Lecture #33
III.16: Signed Measures

§16: Signed Measures. Let μ_1 and μ_2 be measures defined on the same σ-algebra $\mathcal{A}$. If at least of of these measures is finite, then the function

$$\mu(E) = \mu_1(E) - \mu_2(E),\ \ E \in \mathcal{A},$$

is well-defined because $\infty - \infty$ and $-\infty + \infty$ do not occur, and countably additive because for pairwise disjoint $\{E_n\}$ there holds

$$\begin{aligned}\mu\left(\bigcup E_n\right) &= \mu_1\left(\bigcup E_n\right) - \mu_2\left(\bigcup E_n\right)\\ &= \sum \mu_1(E_n) - \sum \mu_2(E_n)\\ &= \sum \big(\mu_1(E_n) - \mu_2(E)\big)\\ &= \sum \mu(E_n).\end{aligned}$$

However, since $\mu(E)$ is not always nonnegative, we call μ a signed measure.

Integration of an integrable function f over $E \in \mathcal{A}$ also generates a signed measure because

$$\int_E f\ d\mu = \int_E f^+\ d\mu - \int_E f^-\ d\mu.$$

Homework Problem 33A. Prove that $E \to \int_E f\ d\mu$ for f integrable is an signed measure.

A signed measure on X is an extended real-valued function μ that satisfies the following

(i) the domain of μ is a σ-algebra in X,
(ii) $\mu(\emptyset) = 0$,
(iii) μ takes at most one of $\pm\infty$,
(iv) μ is countable additive.

Let μ be a signed measure on a σ-algebra $\mathcal{A}$.

A measurable set $E \in \mathcal{A}$ is said to be positive (negative) if for all measurable subsets $A \subset E$ there holds $\mu(A) \geq 0$ ($\mu(A) \leq 0$).

Some basic properties of positive (negative) sets are

(1) the difference of two positive (negative) sets is positive (negative),
(2) the union of two positive (negative) sets is positive (negative),
(3) the countable union of pairwise disjoint positive (negative) sets is positive (negative),
(4) the countable union of positive (negative) sets is positive (negative).

Homework Problem 33B. Prove these four properties for positive sets.

Lemma 16.1. Let μ be a signed measure on a σ-algebra. If $E \in \mathcal{A}$ satisfies $|\mu(E)| < \infty$, then for every measurable $A \subset E$ there holds $|\mu(A)| < \infty$.

Proof. WLOG assume that μ does not take the value ∞.

Let $E \in \mathcal{A}$ satisfy $|\mu(E)| < \infty$.

Let A be a measurable subset of E.

Since μ does not take the value ∞, we have $\mu(A) < \infty$.

If $\mu(A) > 0$, then $|\mu(A)| = \mu(A) < \infty$.

On the other hand, if $\mu(A) < 0$, then since $E = (E - A) \cup A$ disjointly, we have by countable additivity that

$$\mu(E) = \mu(E - A) + \mu(A).$$

If $\mu(A) = -\infty$, then as $\mu(E - A) < \infty$ (μ does not take the value ∞), we have that $\mu(E) = -\infty$, a contradiction to $|\mu(E)| < \infty$.

Hence $\mu(A) \neq -\infty$, so $|\mu(A)| < \infty$. □

Proposition 16.2. Let μ be a signed measure on a σ-algebra $\mathcal{A}$. Every measurable set E of positive finite measure contains a positive subset A of positive measure.

Proof. If E is a positive set, then we take $A = E$.

If E is not a positive set, then E contains a measurable subset of negative measure.

Let n_1 be the smallest positive integer for which there exists $B_1 \subset E$ satisfying

$$\mu(B_1) \leq -\frac{1}{n_1}.$$

If $A_1 = E - B_1$ is a positive set, then we take $A = A_1$.

If A_1 is not a positive set, then it contains a measurable subset of negative measure.

So there exists $B_2 \subset E - B_1$ and the smallest positive integer n_2 such that

$$\mu(B_2) \leq -\frac{1}{n_2}.$$

Proceeding inductively in this fashion, if for some finite positive integer m the set

$$A_m = E - \bigcup_{j=1}^{m} B_j$$

is positive we are done.

If not, the induction produces two sequences of sets $\{B_j\}$ and $\{A_m\}$ and a sequence of positive integers $\{n_j\}$.

We will show that the set

$$A = \bigcap_{m=1}^{\infty} A_m = E - \bigcup_{j=1}^{\infty} B_j$$

is positive by showing that every measurable subset of A has nonnegative measure.

Since $\mu(E)$ is finite, i.e., $|\mu(E)| < \infty$, and A is a measurable subset of E, we have by Lemma 16.1 that $|\mu(A)| < \infty$, i.e., $\mu(A)$ is finite as well.

By construction the sets $\{B_j\}$ and A are measurable and pairwise disjoint and satisfy

$$E = A \cup \bigcup B_j.$$

Since the signed measure μ is countable additive, we have

$$0 < \mu(E) = \mu(A) + \sum_{j=1}^{\infty} \mu(B_j) \leq \mu(A) - \sum_{j=1}^{\infty} \frac{1}{n_j}.$$

Rearranged this gives

$$\sum_{j=1}^{\infty} \frac{1}{n_j} \leq \mu(A) < \infty.$$

The implied convergence of the series implies that $1/n_j \to 0$ as $j \to \infty$.

Because the terms $1/n_j > 0$, the convergent series converges to a positive quantity, implying that $\mu(A) > 0$.

Now let C be a nonempty measurable subset of A.

Then $C \subset A_j$ for all j.

Suppose $\mu(C) < 0$.

In choosing $A_1 = E - B_1$, we looked for the smallest positive integer n_1 for which there is $B_1 \subset E$ satisfying $\mu(B_1) \leq -1/n_1$.

If $\mu(C) \leq -1/n_1$, then we could have chosen $B_1 = C$, implying that $C \cap A_1 = \emptyset$, a contradiction to $C \subset A_1$, whence it must be that

$$\mu(C) > -\frac{1}{n_1} \geq -\frac{1}{n_1 - 1}.$$

In choosing $A_2 = E - (B_1 \cup B_2)$, we looked for the smallest positive integer n_2 for which there is $B_2 \subset E - B_1$ satisfying $\mu(B_2) \leq -1/n_2$.

If $\mu(C) \leq -1/n_2$, then we could have chosen $B_2 = C$, implying that $C \cap A_2 = \emptyset$, a contradiction to $C \subset A_2$, whence it must be that

$$\mu(C) > -\frac{1}{n_2} \geq -\frac{1}{n_2 - 1}.$$

Continuing this argument we see for all $j \in \mathbb{N}$ that

$$\mu(C) > -\frac{1}{n_j - 1}.$$

Since $1/n_j \to 0$ as $j \to \infty$, then $1/(n_j - 1) \to 0$ as $j \to \infty$, and we contradict $\mu(C) < 0$.

Therefore $\mu(C) \geq 0$, and A is a positive set. $\square$