Math 541 Lecture #34
III.17: The Lebesgue-Radon-Nikodym Theorem

§17: The Lebesgue-Radon-Nikodym Theorem. For two measures μ and ν on the same σ-algebra $\mathcal{A}$, we say that ν is absolutely continuous with respect to μ, and write $\nu \ll \mu$, if when $\mu(E) = 0$ for $E \in \mathcal{A}$, there holds $\nu(E) = 0$.

Given a measurable nonnegative $f : X \to \mathbb{R}^*$, the set function

$$E \to \nu(E) = \int_E f \, d\mu, \ E \in \mathcal{A},$$

is a measure on $\mathcal{A}$ that is absolutely continuous with respect to μ.

We ask for the opposite: if $\nu \ll \mu$, is there a measurable nonnegative $f : X \to \mathbb{R}^*$ such that

$$\nu(E) = \int_E f \, d\mu?$$

The partial answer is the content of the Lebesgue-Radon-Nikodym Theorem (Lebesgue proved it for Lebesgue measure on $\mathbb{R}^N$, then Radon extended it to Radon measures, and then Nikodym extended it to general measures).

Theorem 17.1 (Lebesgue-Radon-Nikodym). Let $\{X, \mathcal{A}, \mu\}$ and $\{X, \mathcal{A}, \nu\}$ be σ-finite measure spaces. If $\nu \ll \mu$, then there is a measurable nonnegative function $f : X \to \mathbb{R}^*$ such that

$$\nu(E) = \int_E f \, d\mu, \ E \in \mathcal{A}.$$

The function f is unique up to a set of μ-measure zero.

Some Remarks: (1) The function f here is called the Radon-Nikodym derivative, since formally it satisfies $d\nu = f d\mu$.

(2) The Theorem does not assert that f is μ-integrable. This occurs if and only if ν is finite.

(3) The assumption of σ-finiteness on both measures cannot be removed. You have it as two homework problems to construct counterexamples.

Proof of the Lebesgue-Radon-Nikodym Theorem in the case that both μ and ν are finite measures.

Let Φ be the collection of measurable nonnegative functions $\varphi : X \to \mathbb{R}^*$ that satisfy

$$\int_E \varphi \, d\mu \leq \nu(E) \text{ for all } E \in \mathcal{A}.$$

The collection Φ is nonempty since it contains the zero function.

For two $\varphi_1, \varphi_2 \in \Phi$, the function $\max\{\varphi_1, \varphi_2\}$ also belongs to Φ, because for any $E \in \mathcal{A}$, we have

$$\begin{aligned}\int_E \max\{\varphi_1, \varphi_2\} d\mu &= \int_{E\cap[\varphi_1 \geq \varphi_2]} \varphi_1 \; d\mu + \int_{E\cap[\varphi_1 < \varphi_2]} \varphi_2 \; d\mu \\ &\leq \nu\big(E \cap [\varphi_1 \geq \varphi_2]\big) + \nu\big(E \cap [\varphi_1 < \varphi_2]\big) \\ &= \nu(E).\end{aligned}$$

Since ν is finite, i.e., $\nu(X) < \infty$, the quantity

$$M = \sup_{\varphi \in \Phi} \int_X \varphi \; d\mu \leq \nu(X) < \infty.$$

Let $\{\varphi_n\}$ be a sequence in Φ such that

$$\lim_{n\to\infty} \int_X \varphi_n \; d\mu = M.$$

The sequence of nonnegative measurable functions

$$f_n = \max\{\varphi_1, \dots, \varphi_n\}$$

is nondecreasing and converges pointwise to a measurable nonnegative function $f : X \to \mathbb{R}^*$.

This function f belongs to Φ because by the Monotone Convergence Theorem we have

$$\int_E f \; d\mu = \lim_{n\to\infty} \int_E f_n \; d\mu \leq \nu(E) \text{ for all } E \in \mathcal{A}.$$

To show that this f satisfies $\nu(E) = \int_E f \; d\mu$, we consider the measure

$$\eta(E) = \nu(E) - \int_E f \; d\mu, \;\; E \in \mathcal{A}.$$

If this measure is not the zero measure, then there is $A \in \mathcal{A}$ such that $\eta(A) > 0$.

Since $\nu \ll \mu$, then $\eta \ll \mu$.

Thus $\eta(A) > 0$ implies by absolute continuity with respect to μ that $\mu(A) > 0$ (the contrapositive of absolute continuity of η with respect to μ).

Since μ is finite, i.e., $\mu(X) < \infty$, there exists $\epsilon > 0$ such that

$$\xi(A) = \eta(A) - \epsilon\mu(A) > 0.$$

The function $\xi : \mathcal{A} \to \mathbb{R}^*$ defined by

$$\xi(E) = \eta(E) - \epsilon\mu(E)$$

is a signed measure on $\mathcal{A}$.

By Proposition 16.2, the set A contains a positive subset A_0, so that

$$\xi(E) = \eta(E \cap A_0) - \epsilon\mu(E \cap A_0) \geq 0 \text{ for all } E \in \mathcal{A}.$$

Using the definition of the measure η we have for all $E \in \mathcal{A}$ that

$$\nu(E \cap A_0) - \int_{E \cap A_0} f \; d\mu - \epsilon\mu(E \cap A_0) \geq 0,$$

or rewritten, that for all $E \in \mathcal{A}$ that

$$\int_{E \cap A_0} f \; d\mu + \epsilon\mu(E \cap A_0) \leq \nu(E \cap A_0).$$

This implies that the measurable nonnegative function $f + \epsilon\chi_{A_0}$ belongs to Φ because for all $E \in \mathcal{A}$, we have

$$\begin{aligned}\int_E (f + \epsilon\chi_{A_0}) d\mu &= \int_{E - A_0} f \; d\mu + \int_{E \cap A_0} (f + \epsilon) d\mu \\ &\leq \nu(E - A_0) + \int_{E \cap A_0} f \; d\mu + \epsilon\mu(E \cap A_0) \\ &\leq \nu(E - A_0) + \nu(E \cap A_0) \\ &= \nu(E).\end{aligned}$$

But $f + \epsilon\chi_{A_0} \in \Phi$ contradicts the definition of M because

$$\int_X (f + \epsilon\chi_{A_0}) d\mu = \int_X f \; d\mu + \epsilon \int_X \chi_{A_0} \; d\mu = M + \epsilon\mu(A_0) > M.$$

Thus η is the zero measure, and hence

$$\nu(E) = \int_E f \; d\mu \text{ for all } E \in \mathcal{A}.$$

Suppose $g : X \to \mathbb{R}^*$ is another measurable nonnegative function for which

$$\nu(E) = \int_E g \; d\mu, \text{ for all } E \in \mathcal{A}.$$

To show that $f = g$ a.e. with respect to μ, we consider for $n \in \mathbb{N}$ the sets

$$A_n = \left\{ x \in X : f(x) - g(x) \geq \frac{1}{n} \right\}.$$

Then for all $n \in \mathbb{N}$ we have

$$0 = \nu(A_n) - \nu(A_n) = \int_{A_n} (f - g) d\mu \geq \int_{A_n} \frac{1}{n} \; d\mu = \frac{\mu(A_n)}{n}.$$

These implies that $\mu(A_n) = 0$ so that $f \geq g$ a.e. with respect to μ.

A similar argument shows that $f \leq g$ a.e. with respect to μ, so that $f = g$ a.e. with respect to μ. $\square$